To the Memory of G.I. MARCHUK

Variational Data Assimilation Problems for Sea and Ocean Circulation Models and Methods for Their Solving.

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- Mathematical model of the ocean hydrothermodynamics.
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In INM RAS:

- **1991-2003** – Development of the general methodology
- **2004-2005** – First works on assimilation of sea surface temperature.
- **2005-2006** – Study of problems for semidiscrete models
- **2007** – Existence theorems for "continuous equations"
- **2007** – Methods and technology for solving inverse hydrothermodynamics problem in ocean with variational assimilation of sea surface temperature.
- **2005-2013** – Study of the inverse and variational data assimilation problems ocean dynamics theory for semidiscrete models.

Variational data assimilation problems for general ocean circulation models

- Solvability results for forward problems

- Solvability of inverse and variational data assimilation problems for general models

- Development of the general methodology for solving the problems

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- Splitting methods and semidiscrete models

- Solvability of inverse and variational data assimilation problems for semidiscrete models

- Methodology for solving the problems using the adjoint equation technique

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- Numerical methods for interpolation of the data to be assimilated

- Preparation of sea surface temperature data and data from the ARGO system

- Development of the Data base "World Ocean – INM RAS"

- Development of the Data assimilation system for PC

- Development of the Data assimilation system for system of PC based on Internet technology

Theoretical background of the study

- Mathematical models and complex system theory
- Optimal control theory
- Operator theory and boundary value problems theory
- Iterative algorithms theory
- Adjoint equation theory
- Modern numerical algorithms

1. Mathematical model of the ocean hydrothermodynamics

$$\frac{d\vec{u}}{dt} + \begin{bmatrix} 0 & -f \\ f & 0 \end{bmatrix} \vec{u} - g \cdot grad\xi + A_u \vec{u} + (A_k)^2 \vec{u} = \vec{f} - \frac{1}{\rho_0} grad P_a - \frac{g}{\rho_0} grad \int\limits_0^z \rho_1(T,S) dz',$$

$$\frac{\partial \xi}{\partial t} - m \frac{\partial}{\partial x} (\int\limits_0^H \Theta(z) u dz) - m \frac{\partial}{\partial y} (\int\limits_0^H \Theta(z) \frac{n}{m} v dz) = f_3,$$

$$\frac{dT}{dt} + A_T T = f_T, \ \frac{dS}{dt} + A_S S = f_S,$$

where $\vec{u} = (u, v)$ and $\bar{f} = g \cdot gradG$, $\Theta(z) \equiv \frac{r(z)}{R}$, $\quad r = R - z, \quad 0 < z < H, x \equiv \lambda, y \equiv \theta, n \equiv 1/r, \ m \equiv 1/(r \cos\theta)$. The functions $G, f_3, \xi_0 \equiv \xi$ at $t = 0$ will be "additional unknowns which must be calculated too.

Boundary conditions on the "sea surface" $\Gamma_S \equiv \Omega$ at $z = 0$:

$$\begin{cases} \left(\int\limits_0^H \Theta \vec{u} dz\right) \vec{n} + \beta_0 m_{op} \sqrt{gH}\, \xi = m_{op} \sqrt{gH}\, d_s \text{ on } \partial\Omega, \\ U_n^{(-)} u - \nu \dfrac{\partial u}{\partial z} - k_{33} \dfrac{\partial}{\partial z} A_k u = \tau_x^{(a)} \Big/ \rho_0,\ U_n^{(-)} v - \nu \dfrac{\partial v}{\partial z} - k_{33} \dfrac{\partial}{\partial z} A_k v = \tau_y^{(a)} \Big/ \rho_0, \\ A_k u = 0, \quad A_k v = 0, \\ U_n^{(-)} T - \nu_T \dfrac{\partial T}{\partial z} + \gamma_T (T - T_a) = Q_T + U_n^{(-)} d_T, \\ U_n^{(-)} S - \nu_S \dfrac{\partial S}{\partial z} + \gamma_S (S - S_a) = Q_S + U_n^{(-)} d_S, \end{cases}$$

where
$U_n = \vec{U} \cdot \vec{N}, \vec{U} = (u, v, w) \equiv (\vec{u}, w), \vec{N} = (n_1, n_2, n_3) \equiv (\vec{n}, n_3), U_n^{(-)} = (|U_n| - U_n)/2.$
The boundary function d_T, d_S or Q_T, Q_S can be unknown also.

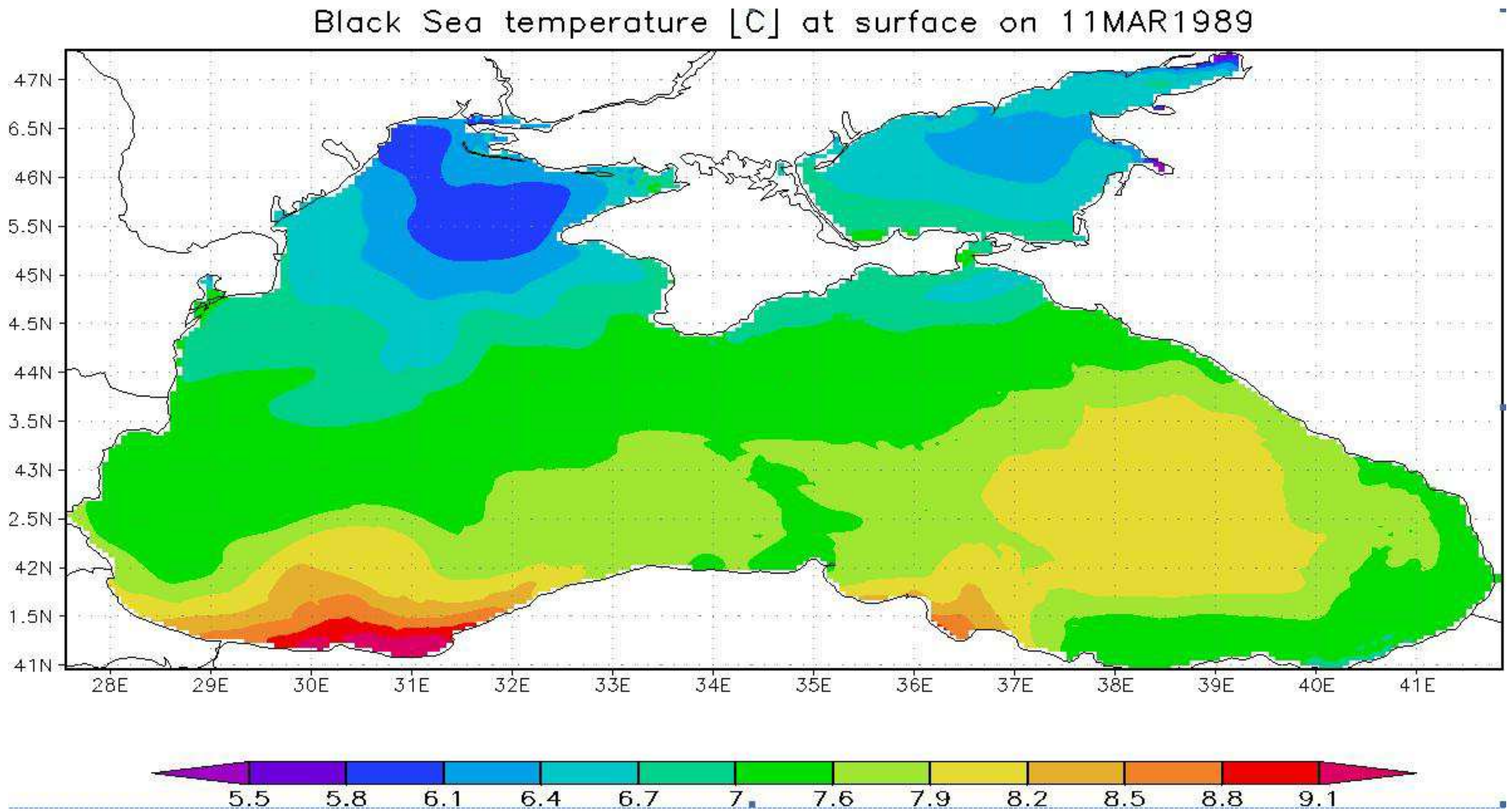
With the function $\phi = (u, v, \xi, T, S)$ known, we calculate

$$w(x,y,z,t) = \frac{1}{r}\left(m\frac{\partial}{\partial x}\left(\int\limits_z^H r u dz'\right) + m\frac{\partial}{\partial y}\left(\frac{n}{m}\int\limits_z^H r v dz'\right)\right), (x,y,t) \in \Omega \times (0, \bar{t}),$$

$$P(x,y,z,t) = P_a(x,y,t) + \rho_0 g(z-\xi) + \int\limits_0^z g\rho_1(T,S)dz'.$$

Note, that for $U_n \equiv \underline{U} \cdot \underline{N}$ (here $U = (u, v, w)$) we always have

$$U_n = 0 \ \text{ on } \ \Gamma_{c,w} \cup \Gamma_H.$$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46.5N
46N
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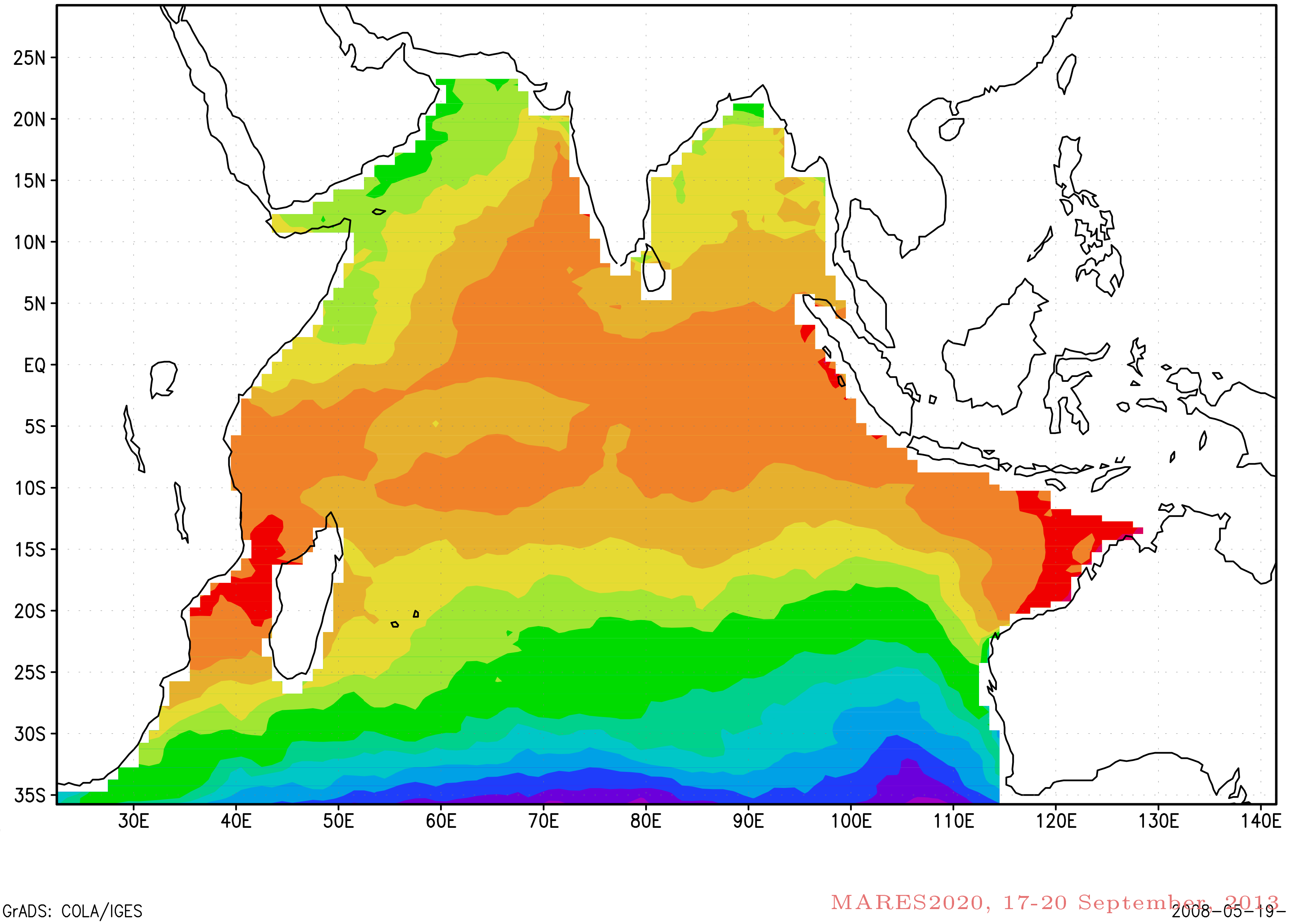
9.1


25N
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60E
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110E
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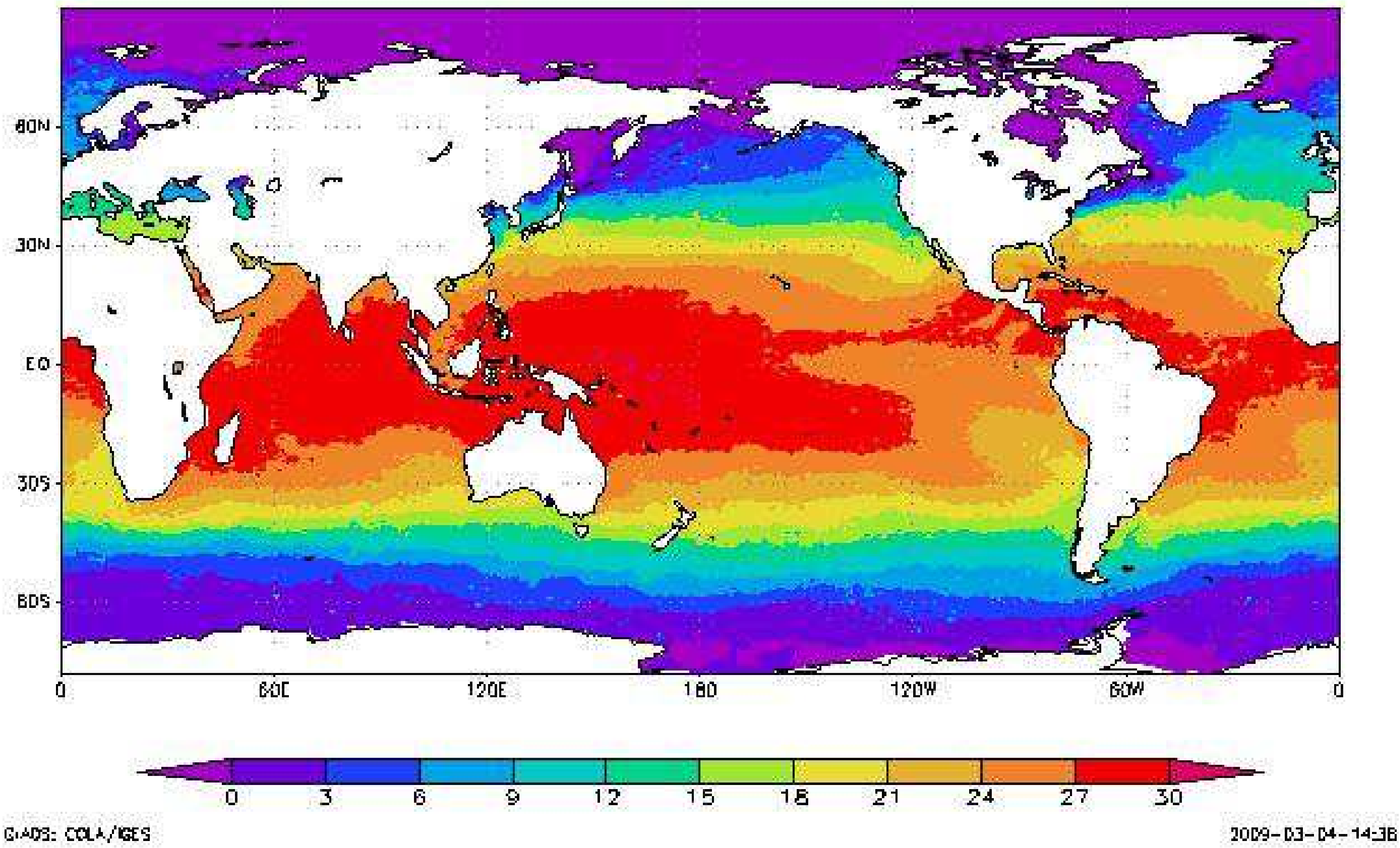

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30N
EQ
30S
60S
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60E
120E
180
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60W
0
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3
6
9
12
15
18
21
24
27
30
GrADS: COLA/IGES
2009-03-04-14:38

2. Approximation by splitting method

- General theory of splitting methods: G.I. Marchuk , N.N. Yanenko, A.A. Samarsky.
- Splitting method in data assimilation: Marchuk G.I., Zalesny V.B. (1993), M. Wenzel, V.B. Zalesny (1996), V.B. Zalesny (2005).
- Studies of inverse and assimilation problems for semidiscrete models in the ocean dynamics: Agoshkov V.I. (2005-2008).
- Studies of class of inverse and data assimilation problems for ocean dynamics models obtained by splitting method: Agoshkov V.I. (2005, 2006), Zalesny V.B. (2008, 2010), Agoshkov V.I., Parmuzin E.I., Zakharova N.B. (2010), Agoshkov V.I., Parmuzin E.I., Shutyaev V.P.(2013).

Problem I

Step 1. We consider the system:

$$\begin{cases} T_t + (\bar{U}, \mathbf{Grad})T - \mathbf{Div}(\hat{a}_T \cdot \mathbf{Grad}\ T) = f_T \text{ in } D \times (t_{j-1}, t_j), \\ T = T_{j-1} \text{ for } t = t_{j-1} \text{ in } D, \\ \bar{U}_n^{(-)} T - \nu_T \dfrac{\partial T}{\partial z} + \gamma_T (T - T_a) = Q_T + \bar{U}_n^{(-)} d_T \text{ on } \Gamma_S \times (t_{j-1}, t_j), \\ \dfrac{\partial T}{\partial N_T} = 0 \text{ on } \Gamma_{w,c} \times (t_{j-1}, t_j), \\ \bar{U}_n^{(-)} T + \dfrac{\partial T}{\partial N_T} = \bar{U}_n^{(-)} d_T + Q_T \text{ on } \Gamma_{w,op} \times (t_{j-1}, t_j), \\ \dfrac{\partial T}{\partial N_T} = 0 \text{ on } \Gamma_H \times (t_{j-1}, t_j), \\ T_j \equiv T \text{ on } D \times (t_{j-1}, t_j), \end{cases}$$

where $\Gamma_w = \Gamma_{w,c} \cup \Gamma_{w,op}$ - the ”vertical lateral boundary”, Γ_H - ”the ocean bottom”.

We consider the subproblem for T in the operator form as

$$(T)_t + LT = \mathcal{F} + BQ, \quad t \in (t_{j-1}, t_j),$$

$$T = T_{j-1}, \quad j = 1, 2, \dots, J,$$

and introduce the additional approximation by the splitting methods:

Step 1.1:

$$(T_1)_t + L_1 T_1 = \mathcal{F}_1, \quad t \in (t_{j-1}, t_j),$$

$$T_1 = T_{j-1} \quad \text{at} \quad t = t_{j-1}$$

Step 1.2:

$$(T_2)_t + L_2 T_2 = \mathcal{F}_2 + BQ, \quad t \in (t_{j-1}, t_j),$$

$$T_2(t_{j-1}) = T_1(t_j).$$

$$T_2(t_j) \equiv T_j \cong T \quad \text{at} \;\; t = t_j.$$

The classical form of the subproblem for $T_2 \equiv T$ is given by:

$$\begin{cases} T_t + \dfrac{1}{2}\left(w_1 \dfrac{\partial T}{\partial z} + \dfrac{1}{r^2}\dfrac{\partial (r^2 w_1 T)}{\partial z} \right) - \dfrac{1}{r^2}\dfrac{\partial}{\partial z} r^2 \nu_T \dfrac{\partial T}{\partial z} = f_T \text{ in } D \text{ at } t \in (t_{j-1}, t_j), \\ T = T_1(t_j) \quad \text{at } t = t_{j-1}, \\ -\nu_T \dfrac{\partial T}{\partial z} = Q \text{ at } z = 0, \\ \nu_T \dfrac{\partial T}{\partial z} = 0 \text{ at } z = H, \end{cases}$$

where

$$\bar{U}_n^{(-)} = \frac{|\bar{U}_n| - \bar{U}_n}{2} = \frac{1}{2}(|\bar{w}_1| + \bar{w}_1) = \frac{1}{2}(|\bar{w}| + \bar{w}) \text{ at } z = 0,$$

$$Q \equiv Q_T - \gamma_T (T - T_a) - \bar{U}_n^{(-)} T + \bar{U}_n^{(-)} d_T .$$

Step 2.

$$
\begin{cases}
S_t + (\bar{U}, \mathbf{Grad})S - \mathbf{Div}(\hat{a}_S \cdot \mathbf{Grad}\ S) = f_S \text{ in } D \times (t_{j-1}, t_j), \\
S = S_{j-1} \text{ at } t = t_{j-1} \text{ in } D, \\
\bar{U}_n^{(-)} S - \nu_S \dfrac{\partial S}{\partial z} + \gamma_S (S - S_a) = Q_S + \bar{U}_n^{(-)} d_S \text{ on } \Gamma_S \times (t_{j-1}, t_j), \\
\dfrac{\partial S}{\partial N_S} = 0 \text{ on } \Gamma_{w,c} \times (t_{j-1}, t_j), \\
\bar{U}_n^{(-)} S + \dfrac{\partial S}{\partial N_S} = \bar{U}_n^{(-)} d_S + Q_S \text{ on } \Gamma_{w,op} \times (t_{j-1}, t_j), \\
\dfrac{\partial S}{\partial N_S} = 0 \text{ on } \Gamma_H \times (t_{j-1}, t_j), \\
S_j \equiv S \text{ on } D \times (t_{j-1}, t_j).
\end{cases}
$$

We rewrite the subproblem for S in the operator form as

$$(S)_t + LS = \mathcal{F} + BQ, \quad t \in (t_{j-1}, t_j),$$

$$S = S_{j-1}, \quad j = 1, 2, \ldots, J,$$

and introduce the additional approximation by the splitting methods:
Step 1.1:

$$(S_1)_t + L_1 S_1 = \mathcal{F}_1, \quad t \in (t_{j-1}, t_j),$$

$$S_1 = S_{j-1} \quad \text{at} \quad t = t_{j-1}$$

Step 1.2:

$$(S_2)_t + L_2 S_2 = \mathcal{F}_2 + BQ, \quad t \in (t_{j-1}, t_j),$$

$$S_2(t_{j-1}) = S_1(t_j).$$

$$S_2(t_j) \equiv S_j \cong S \quad \text{at} \;\; t = t_j.$$

The classical form of the subproblem for $S_2 \equiv S$ is given by:

$$\begin{cases} S_t + \frac{1}{2}\left(w_1 \frac{\partial S}{\partial z} + \frac{1}{r^2}\frac{\partial (r^2 w_1 S)}{\partial z}\right) - \frac{1}{r^2}\frac{\partial}{\partial z} r^2 \nu_S \frac{\partial S}{\partial z} = f_S \;\text{ in } D \;\text{ at }\; t \in (t_{j-1}, t_j), \\ S = S_1(t_j) \quad \text{at} \;\; t = t_{j-1}, \\ -\nu_S \frac{\partial S}{\partial z} = Q \;\text{ at }\; z = 0, \\ \nu_S \frac{\partial S}{\partial z} = 0 \;\text{ at }\; z = H, \end{cases}$$

where

$$\bar{U}_n^{(-)} = \frac{|\bar{U}_n| - \bar{U}_n}{2} = \frac{1}{2}(|\bar{w}_1| + \bar{w}_1) = \frac{1}{2}(|\bar{w}| + \bar{w}) \;\text{ at }\; z = 0,$$

$$Q \equiv Q_S - \gamma_S (S - S_a) - \bar{U}_n^{(-)} S + \bar{U}_n^{(-)} d_S .$$

Step 3: The subproblems for the velocity components

Step 3.1

$$\begin{cases} \underline{u}_t^{(1)} + \begin{bmatrix} 0 & -\ell \\ \ell & 0 \end{bmatrix} \underline{u}^{(1)} - g \cdot \mathbf{grad}\xi = g \cdot \mathbf{grad}G - \frac{1}{\rho_0}\mathbf{grad}\left(P_a + g\int\limits_0^z \rho_1(\bar{T},\bar{S})dz'\right) \\ \text{ in } D \times (t_{j-1}, t_j), \\ \xi_t - \mathbf{div}\left(\int\limits_0^H \Theta\underline{u}^{(1)}dz\right) = f_3 \text{ in } \Omega \times (t_{j-1}, t_j), \\ \underline{u}^{(1)} = \underline{u}_{j-1},\ \xi = \xi_{j-1} \text{ at } t = t_{j-1}, \\ \left(\int\limits_0^H \Theta\underline{u}^{(1)}dz\right) \cdot n + \beta_0 m_{op}\sqrt{gH}\xi = m_{op}\sqrt{gH}d_s \text{ on } \partial\Omega \times (t_{j-1}, t_j), \\ \underline{u}_j^{(1)} \equiv \underline{u}^{(1)}(t_j) \text{ in } D \end{cases}$$

If we write down $\underline{u}^{(1)}$ in the following form: $\underline{u}^{(1)} = \underline{U}^{(1)}(\lambda,\theta,t) + \underline{u}'(\lambda,\theta,z,t)$ where

$$\underline{U}^{(1)} = \frac{1}{H_1}\int_0^H \Theta\underline{u}^{(1)}dz, \qquad H_1 = \int_0^H \Theta dz,$$

then Step 3.1 is reduced to two subproblems for the functions $\underline{U}^{(1)}, \underline{u}_1'$.

Step 3.1

First of them is "The ocean tide theory problem":

$$\left\{ \begin{array}{l} \underline{U}_t^{(1)} + \begin{bmatrix} 0 & -\ell \\ \ell & 0 \end{bmatrix} \underline{U}^{(1)} - g\,\mathrm{grad}\,\xi = g\,\mathrm{grad}\,G - \underline{I} \quad \text{in} \quad D \times (t_{j-1}, t_j) \\ \xi_t - \mathrm{div}(H_1 \underline{U}^{(1)}) = f_3 \quad \text{in} \quad \Omega \times (t_{j-1}, t_j) \\ \underline{U}^{(1)}(t_{j-1}) = \frac{1}{H_1} \int_0^H \Theta \underline{u}_{j-1} dz, \qquad \xi(t_{j-1}) = \xi_{j-1} \quad \text{in } \Omega \\ \\ (H_1 \underline{U}^{(1)}) \cdot \underline{n} + \beta_0 m_{\mathrm{op}} \sqrt{gH} \xi = m_{\mathrm{op}} \sqrt{gH} d_s \end{array} \right.$$

where

$$\underline{I} = (I_\lambda, I_\theta) = \frac{1}{\rho_0} \Big(\mathrm{grad}\, P_a + g \frac{1}{H_1} \int_0^H \Theta dz \int_0^z \mathrm{grad}\, \rho_1(\bar{T}, \bar{S}) dz' \Big).$$

The study and solution of this subproblem and its adjoint problem have the crusial meaning for one of the inverse and data assimilation problems studied.

The second subproblem is :

$$\begin{cases} (\underline{u}_1')_t + \begin{bmatrix} 0 & -\ell \\ \ell & 0 \end{bmatrix} \underline{u}_1' = \frac{g}{\rho_0}\Big(\frac{1}{H_1} \int_0^H \Theta dz \int_0^z \text{grad}\, \rho_1(\bar{T}, \bar{S}) dz' \\ - \int_0^z \text{grad}\, \rho_1(\bar{T}, \bar{S}) dz' \Big) \\ \underline{u}_1'(t_{j-1}) = \underline{u}_{j-1} - \dfrac{1}{H_1} \displaystyle\int_0^H \Theta u_{j-1} dz \end{cases}$$

Step 3.2

$$\begin{cases} \underline{u}_t^{(2)} + \begin{bmatrix} 0 & -f_1(\bar{u}) \\ f_1(\bar{u}) & 0 \end{bmatrix} \underline{u}^{(2)} = 0 \text{ in } D \times (t_{j-1}, t_j), \\ \underline{u}^{(2)} = \underline{u}_j^{(1)} \text{ при } t = t_{j-1} \text{ in } D, \\ \underline{u}_j^{(2)} \equiv \underline{u}^{(2)}(t_j) \text{ in } D, \end{cases}$$

Step 3.3

$$
\begin{cases}
\underline{u}_t^{(3)} + (\bar{U}, \mathbf{Grad})\underline{u}^{(3)} - \mathbf{Div}(\hat{a}_u \cdot \mathbf{Grad}\,)\underline{u}^{(3)} + (A_k)^2\underline{u}^{(3)} = 0 \text{ in } D \times (t_{j-1}, t_j), \\
\underline{u}^{(3)} = \underline{u}^{(2)} \text{ at } t = t_{j-1} \text{ in } D, \\
\bar{U}_n^{(-)}\underline{u}^{(3)} - \nu_u \frac{\partial \underline{u}^{(3)}}{\partial z} - k_{33}\frac{\partial}{\partial z}(A_k\underline{u}^{(3)}) = \frac{\underline{\tau}^{(a)}}{\rho_0}, A_k\underline{u}^{(3)} = 0 \text{ on } \Gamma_S \times (t_{j-1}, t_j), \\
U_n^{(3)} = 0, \frac{\partial U^{(3)}}{\partial N_u} \cdot \bar{\underline{\tau}}_w + \left(\frac{\partial}{\partial N_k} A_k\underline{u}^{(3)}\right) \cdot \underline{\tau}_w = 0, A_k\underline{u}^{(3)} = 0 \text{ on } \Gamma_{w,c} \times (t_{j-1}, t_j), \\
\bar{U}_n^{(-)}(\tilde{U}^{(3)} \cdot \underline{N}) + \frac{\partial \tilde{U}^{(3)}}{\partial N_u} \cdot \bar{N} + \left(\frac{\partial}{\partial N_k} A_k\underline{u}^{(3)}\right) \cdot \bar{N} = \bar{U}_n^{(-)} d, A_k\underline{u}^{(3)} = 0 \text{on } \Gamma_{w,op} \times (t_{j-1}, t_j) \\
\bar{U}_n^{(-)}(\tilde{U}^{(3)} \cdot \bar{\tau}_w) + \frac{\partial \tilde{U}^{(3)}}{\partial N_u} \cdot \bar{\tau}_w + \left(\frac{\partial}{\partial N_k} A_k\underline{u}^{(3)}\right) \cdot \underline{\tau}_w = 0, A_k\underline{u}^{(3)} = 0 \text{ on } \Gamma_{w,op} \times (t_{j-1}, t_j), \\
\frac{\partial \underline{u}^{(3)}}{\partial N_u} = \frac{\tau^{(b)}}{\rho_0} \text{ on } \Gamma_H \times (t_{j-1}, t_j),
\end{cases}
$$

where

$$
\underline{u}^{(3)} = (u^{(3)}, v^{(3)}),\ \tau^{(a)} = (\tau_x^{(a)}, \tau_y^{(a)}),
$$
$$
U^{(3)} = (u^{(3)}, w^{(3)}(u^{(3)}, v^{(3)})),\ \tilde{U}^{(3)} = (u^{(3)}, 0),\ \ \tau^{(b)} = (\tau_x^{(b)}, \tau_y^{(b)}).
$$

3. Inverse and variational data assimilation problems, I

Let us assume, that the unique function which is obtained by observation data processing is the function ξ_{obs} on $\bar{\Omega} \equiv \Omega \cup \partial\Omega$ at $t \in (t_{j-1}, t_j)$, $j = 1, 2, \ldots, J$. Let by physical meaning this function is an approximation to sea level function ξ on Ω, i.e on the boundary, when $z = 0$. We permit that the function ξ_{obs} is known only on the part of $\Omega \times (0, \bar{t})$ and we define a support of this function as m_0. Beyond of this area we suppose function ξ_{obs} is trivial.

Let the functions G, f_3, ξ_0 are "additional unknown functions" and we state the following inverse problem - **Problem Inv 1**: *find the solution* $\phi = (u, v, \xi, T, S)$ *of the Problem* I *and functions* G, f_3, ξ_0, *such that,* $m_0(\xi - \xi_{obs}) = 0$.

To study this inverse problem we apply general methodology for solving data assimilation problems (Agoshkov V., 2003) and classical results of the inverse problem theory (A.N. Tikhonov, M.M. Lavrentiev, V.K. Ivanov, V.V. Vasin, V.G. Romanov, M.V. Klibanov,Yu.E. Anikonov,S.I. Kabanikhin, A.Hasanov, V.G. Yakhno).

Introduce the cost functional $\Im_\alpha$ of the form:

$$\Im_\alpha \equiv \Im_\alpha(\xi_0, G, f_3, \Phi) = \frac{1}{2}\Big\{\alpha_0 \bar{t} \left\|\xi_0 - \xi^{(0)}\right\|^2_{L_2(g;\Omega)} + \alpha_f \left\|f_3 - f_3^{(0)}\right\|^2_{L_2(0,\bar{t};L_2(g;\Omega))}$$

$$+\alpha_G \left\|G - G^{(0)}\right\|^2_{L_2(0,\bar{t};L_2(g;\Omega))} \Big\} + \Im_0(\Phi) = \sum_{j=1}^{J} \int_{t_{j-1}}^{t_j} \Im_\alpha^{(j)} dt,$$

where

$$\Im_0(\Phi) \equiv \Im_0(\xi) = \frac{1}{2} \left\|m_0(\xi - \xi_{\text{obs}})\right\|^2_{L_2(0,\bar{t};L_2(g;\Omega))}$$

$$\Im_\alpha^{(j)} = \frac{1}{2}\Big\{\alpha_0 \Delta t_j \left\|\xi_0 - \xi^{(0)}\right\|^2_{L_2(g;\Omega)} + \alpha_f \left\|f_3 - f_3^{(0)}\right\|^2_{L_2(g;\Omega)}(t)$$

$$+\alpha_G \left\|G - G^{(0)}\right\|^2_{L_2(g;\Omega)}(t) + \left\|m_0(\xi - \xi_{\text{obs}})\right\|^2_{L_2(g;\Omega)}(t)\Big\}.$$

Here $\alpha \equiv (\alpha_0, \alpha_f, \alpha_G)$, $\alpha_0 \geq 0$, $\alpha_f \geq 0$, $\alpha_G \geq 0$ are regularization parameters that may be dimensional values. Furthermore, it is possible to specify α_f, α_G depending on $\alpha_0 \geq 0$, (for instance, $\alpha_G = \alpha_0$, $\alpha_f = \alpha_0 \bar{t}^2$, etc.).
We can formulate the data assimilation problem - **Problem A 1**: *find the solution* ϕ *of the Problem* I *and function* G, f_3, ξ_0, *such that, the cost functional is minimal on the set of the solutions.*

Let us consider the problem on the first time step (t_0, t_1). Then the optimality conditions are:

$$\begin{cases} t_1 \alpha_0 (\xi_0 - \xi^{(0)}) + \xi^*(t_0) = 0 \quad \text{in } \Omega \\ \alpha_f (f_3 - f_3^{(0)}) + \xi^* = 0 \quad \text{in } \Omega \times (t_0, t_1) \\ \alpha_G (G - G^{(0)}) - \operatorname{div}\Big(\int_0^{\mathrm{H}} \Theta \underline{u}_1^* dz \Big) = 0 \quad \text{in } \Omega \times (t_0, t_1), \end{cases}$$

where $\xi^*, \underline{u}_1^*$ are the solution of the adjoint problem:

$$\begin{cases} -(\underline{u}_1^*)_t - \begin{bmatrix} 0 & -\ell \\ \ell & 0 \end{bmatrix} \underline{u}_1^* + g \operatorname{grad} \xi^* = 0 \quad \text{in } D \times (t_0, t_1) \\ -\xi_t^* + \operatorname{div}\Big(\int_0^H \Theta \underline{u}_1^* dz \Big) = m_0 (\xi - \xi_{\mathrm{obs}}) \quad \text{in } \Omega \times (t_0, t_1) \\ -\Big(\int_0^H \Theta \underline{u}_1^* dz \Big) \cdot \underline{n} + \beta_0 m_{\mathrm{op}} \sqrt{gH} \xi^* = 0 \quad \text{on } \partial\Omega \times (t_0, t_1) \\ \xi^* = 0, \qquad \underline{u}_1^* = 0 \quad \text{at } t = t_1 \end{cases}$$

(or here $\xi^* = m_0 (\xi - \xi_{\mathrm{obs}})(t_1)$ at $t = t_1$).

Definition : Problem Inv 1 is densely solvable if for any $\epsilon > 0$ there is a solution ϕ of the Problem I such that $\Im_0(\phi) < \epsilon$.

Proposition 1. If $\text{supp}(\xi_{\text{obs}}) = \bar{\Omega} \times [t_0, t_1]$ and $(G, f_3)_{L_2(g;\Omega)} = 0 \ \forall t$ then Problem Inv1 is uniquely and densely solvable. The solution of Problem A1 can be taken as an approximate solution of Problem Inv1 for sufficiently small α.

Proposition 2. If $\text{mes}(\partial\Omega \cap \Gamma_{w,\text{op}}) > 0$ and the function G is sought additionally only then Problem Inv 1 is densely solvable.

Proposition 3. If $\text{mes}(\text{supp}(\xi_{\text{obs}})) > 0$ and the function f_3 is sought additionally only then Problem Inv 1 is densely solvable.

Iterative process

For numerical implementation of the algorithm of solving the whole problem in (t_0, t_1) it is sufficient to solve two initial-boundary problems for parabolic equations (after that T and S will be defined in $D \times (t_0, t_1)$) and carry out Step 3 including the data assimilation block. A numerical solution of the problem at Step 3 can be obtained by the following iterative algorithm: if $f_3^{(k)}, G^{(k)}, \xi_0^{(k)}$ are defined, we solve the subproblems from the Step 3 for $\xi_0 = \xi_0^{(k)}$, $f_3 = f_3^{(k)}$, $G = G^{(k)}$ and then solve adjoint problem and compute the new approximation $f_3^{(k+1)}, G^{(k+1)}, \xi_0^{(k+1)}$.

$$\begin{cases} \xi_0^{(k+1)} = \xi_0^{(k)} - \gamma_k(\alpha_0(\xi_0^{(k)} - \xi^{(0)}) + \xi^*(t_0)) \quad \text{in } \Omega \\ f_3^{(k+1)} = f_3^{(k)} - \gamma_k(\alpha_f(f_3^{(k)} - f_3^{(0)}) + \xi^*) \quad \text{in } \Omega \times (t_0, t_1) \\ G^{(k+1)} = G^{(k)} - \gamma_k\Big(\alpha_G(G^{(k)} - G^{(0)}) - \mathrm{div}\Big(\int_0^H \Theta u_1^* dz\Big)\Big) \\ \qquad \text{in } \Omega \times (t_0, t_1) \\ \text{provided that } \displaystyle\int_\Omega \mathrm{G}^{(i)} d\Omega = 0 \quad \forall i. \end{cases}$$

In the case of an appropriate selection of parameters $\{\gamma_k\}$ the iterative process converges. In virtue of the compact solvability property, the following values can be assumed an efficient choice of γ_k:

$$\gamma_k = \frac{\dfrac{1}{2}\displaystyle\int_{t_0}^{t_1}\int_\Omega m_0(\xi^{(k)} - \xi_{\mathrm{obs}})^2 d\Omega dt}{\Big(\displaystyle\int_\Omega (\xi^*(t_0))^2 d\Omega + \int_{t_0}^{t_1}\int_\Omega (\xi^*)^2 d\Omega dt + \int_{t_0}^{t_1}\int_\Omega \Big(\mathrm{div}\int_0^H \underline{u}_1^{*(k)} \Theta dz\Big)^2 d\Omega dt\Big)}.$$

After the criterion of stopping the iteration process is satisfied, it is necessary to used the computed $f_3^{(k+1)}, G^{(k+1)}, \xi_0^{(k+1)}$ to solve other subproblems from the Step 3 and to obtain an approximate solution to the whole problem in $D \times (t_0, t_1)$.
After solving all problems and implementing the iteration process in (t_0, t_1), the variation assimilation problem is solved similarly in the subsequent intervals $(t_{j-1}, t_j), j = 2, 3, \ldots$. **In view of the established properties of unique and dense solvability of the considered Problem Inv and data assimilation problem in each time interval, we can state that the system of all approximate solutions $\{\phi_j\}$ imparts the minimal value of whole cost functional, i.e., is the solution to the considered problem for the whole interval $(0, \bar{t}\,)$.**

4. Inverse and variational data assimilation problems, II

Problem Inv 2

Assume that the sea surface temperature (SST), observed on a subset $\Omega^{(j)}$ of Ω, is denoted by $T_{obs} \equiv T_{obs}^{(j)}$ when $t \in (t_{j-1}, t_j), m_0^{(j)}$ is the characteristic function of this subset $(j = 1, 2, ..., J)$. Considering the boundary condition for T at $z = 0$ we write it in the following form:

$$-\nu_T \frac{\partial T}{\partial z} = Q \; at \; z = 0 \; on \; \Omega^{(j)} \times (t_{j-1}, t_j),$$

$$\bar{U}_n^{(-)} T - \nu_T \frac{\partial T}{\partial z} + \gamma_T (T - T_a) = Q_T + \bar{U}_n^{(-)} d_T \; at \;\; z = 0 \; on \; (\Omega \backslash \Omega^{(j)}) \times (t_{j-1}, t_j),$$

where the function $Q \equiv Q^{(j)} (j = 1, 2, ..., J)$.
Let the functions $Q^{(j)}$ are "additional unknown functions" and we state the following inverse problem - **Problem Inv 2**: *find the solution* $\phi = (u, v, \xi, T, S)$ *of the Problem* I *and functions* $Q^{(j)}$, *such that,* $m_0^{(j)} (T - T_{obs}^{(j)}) = 0$ on $\Omega, j = 1, 2, ..., J$.

Introduce the cost functional $\Im_\alpha$ of the form:

$$\Im_\alpha \equiv \Im_\alpha(Q,\Phi) = \frac{1}{2}\int\limits_0^{\bar t}\int\limits_\Omega \alpha|Q - Q^{(0)}|^2 d\Omega dt + \Im_0(\Phi) = \sum_{j=1}^{J}\int\limits_{t_{j-1}}^{t_j} \Im_\alpha^{(j)} dt,$$

where

$$\Im_0(\Phi) \equiv \Im_0(Q) = \frac{1}{2}\int\limits_0^{\bar t}\int\limits_{\Omega_0(t)} m_0|T - T_{obs}|^2 d\Omega dt,$$

$$\Im_\alpha^{(j)} = \frac{1}{2}\int\limits_{t_{j-1}}^{t_j}\int\limits_{\Omega^{(j)}} \alpha|Q - Q^{(0)}|^2 d\Omega dt + \frac{1}{2}\int\limits_{t_{j-1}}^{t_j}\int\limits_{\Omega^{(j)}} m_0^{(j)}|T - T_{obs}^{(j)}|^2 d\Omega dt.$$

Here $\alpha \geq 0$, is a "regularization" or "penalty" function, that may be constant.

Data assimilation problem - **Problem A 2**: *find the solution* ϕ *of the Problem* I *and functions* $\{Q^{(j)}\}$, *such that, the cost functional is minimal on the set of the solutions.*

The optimality system obtained consist of successive solving the variational assimilation problem on intervals $t \in (t_{j-1}, t_j)$, $j = 1, 2, \dots, J$. The method can be discribed as follows:

STEP 1. We solve system of equations, which arise from minimization of the functional J_α on the set of the solution of the equations. This system consists of equations for T_1, T_2, Q and system of adjoint equations:

$$\begin{cases} -(T_2^*)_t + L_2^* T_2^* = B^* m_0^{(1)} (T - T_{obs}^{(1)}) \quad \text{in} \;\; D \times (t_0, t_1), \\ T_2^* = 0 \quad \text{for} \;\; t = t_1, \end{cases}$$

$$\begin{cases} -(T_1^*)_t + L_1^* T_1^* = 0 \quad \text{in} \;\; D \times (t_0, t_1), \\ T_1^* = T_2^*(t_0) \quad \text{for} \;\; t = t_1 \end{cases}$$

$$\alpha(Q - Q^{(0)}) + T_2^* = 0 \quad \text{on} \;\; \Omega_0^{(1)} \times (t_0, t_1).$$

Functions T_2, $Q(t_1)$ are accepted as approximations to functions T, Q of the full solution for the Problem I at $t > t_1$, and $T_2(t_1) \cong T(t_1)$ is taken as an initial condition to solve the problem on the interval (t_1, t_2).

STEP 2. Solve problem for S:

$$S_t + (\bar{U}, \mathbf{Grad}) S - \mathbf{Div}(\hat{a}_S \cdot \mathbf{Grad}\ S) = f_S \text{ in } D \times (t_0, t_1)$$

with corresponding boundary and initial conditions. After that the function S is accepted as an approximate solution, and the function $S(t_1)$ is taken as an initial condition for the problem for the interval (t_1, t_2).

STEP 3. Solve equations of the velocity module.

Iterative process

Given $Q^{(k)}$ one solve all subproblems from step 1, adjoint problem for this step and define new correction $Q^{(k+1)}$

$$Q^{(k+1)} = Q^{(k)} - \gamma_k^{(j)}(\alpha(Q^{(k)} - Q^{(0)}) + T_2^*) \quad \text{on} \;\; \Omega_0^{(j)} \times (t_{j-1}, t_j).$$

Parameters $\{\gamma_k\}$ can be calculated at $\alpha \approx +0$, by the property of dense solvability, as:

$$\gamma_k^{(j)} = \frac{1}{2} \frac{\int\limits_{t_{j-1}}^{t_j} \int\limits_{\Omega_0^{(j)}} (T - T_{obs}^{(j)})^2 \Big|_{\sigma=0} d\Omega dt}{\int\limits_{t_{j-1}}^{t_j} \int\limits_{\Omega_0^{(j)}} (T_2^*)^2 \Big|_{\sigma=0} d\Omega dt}.$$

In view of the established properties of unique and dense solvability of the considered Problem Inv and data assimilation problem in each time interval, we can state that the system of all approximate solutions $\{\phi_j\}$ imparts the minimal value of whole cost functional, i.e., is the solution to the considered problem for the whole interval $(0, \bar{t}\,)$.

Definition : Problem Inv 2 is densely solvable if for any $\epsilon > 0$ there is a solution ϕ of the Problem I such that $\Im_0(\phi) < \epsilon$.

Proposition 1. Problem Inv 2 is uniquely and densely solvable. The solution of Problem A2 can be taken as an approximate solution of Problem Inv 2 for sufficiently small α.

5. Inverse and variational data assimilation problems, III

Problem Inv 3

Assume that the sea surface salinity (SSS), observed on a subset $\Omega^{(j)}$ of Ω, is denoted by $S_{obs} \equiv S_{obs}^{(j)}$ when $t \in (t_{j-1}, t_j), m_0^{(j)}$ is the characteristic function of this subset $(j = 1, 2, ..., J)$. Considering the boundary condition for S at $z = 0$ we write as:

$$-\nu_S \frac{\partial S}{\partial z} = Q \ at \ z = 0 \ on \ \Omega^{(j)} \times (t_{j-1}, t_j),$$

$$\bar{U}_n^{(-)} S - \nu_S \frac{\partial S}{\partial z} + \gamma_S (S - S_a) = Q_S + \bar{U}_n^{(-)} d_S \ at \ \ z = 0 \ on \ (\Omega \backslash \Omega^{(j)}) \times (t_{j-1}, t_j),$$

where the function $Q \equiv Q^{(j)} (j = 1, 2, ..., J)$.

Let the functions $Q^{(j)}$ are "additional unknown functions" and we state the following inverse problem - **Problem Inv 3**: *find the solution* $\phi = (u, v, \xi, T, S)$ *of the Problem* I *and functions* $Q^{(j)}$, *such that,* $m_0^{(j)} (S - S_{obs}^{(j)}) = 0$ on $\Omega, j = 1, 2, ..., J$.

Variational approach for solving the inverse problem

Introduce the cost functional $\Im_\alpha$ of the form:

$$\Im_\alpha \equiv \Im_\alpha(Q,\Phi) = \frac{1}{2}\int_0^{\bar{t}}\int_\Omega \alpha|Q - Q^{(0)}|^2 d\Omega dt + \Im_0(\Phi) = \sum_{j=1}^{J}\int_{t_{j-1}}^{t_j} \Im_\alpha^{(j)} dt,$$

where

$$\Im_0(\Phi) \equiv \Im_0(Q) = \frac{1}{2}\int_0^{\bar{t}}\int_{\Omega_0(t)} m_0|S - S_{obs}|^2 d\Omega dt,$$

$$\Im_\alpha^{(j)} = \frac{1}{2}\int_{t_{j-1}}^{t_j}\int_{\Omega^{(j)}} \alpha|Q - Q^{(0)}|^2 d\Omega dt + \frac{1}{2}\int_{t_{j-1}}^{t_j}\int_{\Omega^{(j)}} m_0^{(j)}|S - S_{obs}^{(j)}|^2 d\Omega dt.$$

Here $\alpha \geq 0$, is a "regularization"or "penalty"function ,that may be constant.
Data assimilation problem - **Problem A 3**: *find the solution* ϕ *of the Problem* I *and functions* $\{Q^{(j)}\}$, *such that, the cost functional is minimal on the set of the solutions.*
This problem for the case $\Omega^{(j)} \equiv \Omega (j = 1, 2, ..., J)$ has been studied and numerically solved by Agoshkov V.I., Parmuzin E.I. and Shutyaev V.P.[2008].

Definition : Problem Inv 3 is densely solvable if for any $\epsilon > 0$ there is a solution ϕ of the Problem I such that $\Im_0(\phi) < \epsilon$.

Proposition 1. Problem Inv 3 is uniquely and densely solvable. The solution of Problem A3 can be taken as an approximate solution of Problem Inv 3 for sufficiently small α.

The optimality system obtained consist of successive solving the variational assimilation problem on intervals $t \in (t_{j-1}, t_j)$, $j = 1, 2, \dots, J$. The method can be discribed as follows:

STEP 1. Solve problem for T:

$$T_t + (\bar{U}, \mathbf{Grad})T - \mathbf{Div}(\hat{a}_T \cdot \mathbf{Grad}\ T) = f_T \text{ in } D \times (t_0, t_1)$$

with corresponding boundary and initial conditions. After that the function T is accepted as an approximate solution, and the function $T(t_1)$ is taken as an initial condition for the problem for the interval (t_1, t_2).

STEP 2. We solve system of equations, which arise from minimization of the functional J_α on the set of the solution of the equations. This system consists of equations for S_1, S_2, Q and system of adjoint equations:

$$\begin{cases} -(S_2^*)_t + L_2^* S_2^* = B^* m_0^{(1)}(S - S_{obs}^{(1)}) \quad \text{in} \ \ D \times (t_0, t_1), \\ S_2^* = 0 \quad \text{for} \ \ t = t_1, \end{cases}$$

$$\begin{cases} -(S_1^*)_t + L_1^* S_1^* = 0 \quad \text{in} \ \ D \times (t_0, t_1), \\ S_1^* = S_2^*(t_0) \quad \text{for} \ \ t = t_1 \end{cases}$$

$$\alpha(Q - Q^{(0)}) + S_2^* = 0 \quad \text{on} \ \ \Omega_0^{(1)} \times (t_0, t_1).$$

Functions S_2, $Q(t_1)$ are accepted as approximations to functions S, Q of the full solution for the Problem I at $t > t_1$, and $S_2(t_1) \cong S(t_1)$ is taken as an initial condition to solve the problem on the interval (t_1, t_2).

STEP 3. Solve equations of the velocity module.

Given $Q^{(k)}$ one solve all subproblems from step 1, adjoint problem for this step and define new correction $Q^{(k+1)}$

$$Q^{(k+1)} = Q^{(k)} - \gamma_k^{(j)}(\alpha(Q^{(k)} - Q^{(0)}) + S_2^*) \quad \text{on} \;\; \Omega_0^{(j)} \times (t_{j-1}, t_j).$$

Parameters $\{\gamma_k\}$ can be calculated at $\alpha \approx +0$, by the property of dense solvability, as:

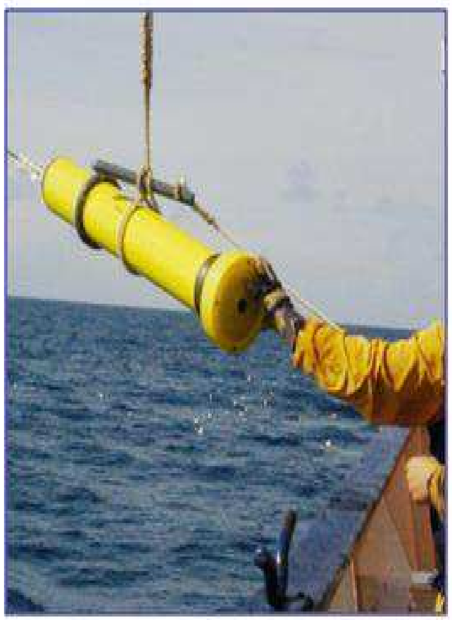
$$\gamma_k^{(j)} = \frac{1}{2} \, \frac{\int\limits_{t_{j-1}}^{t_j} \int\limits_{\Omega_0^{(j)}} (S - S_{obs}^{(j)})^2 \Big|_{\sigma=0} d\Omega dt}{\int\limits_{t_{j-1}}^{t_j} \int\limits_{\Omega_0^{(j)}} (S_2^*)^2 \Big|_{\sigma=0} d\Omega dt}.$$

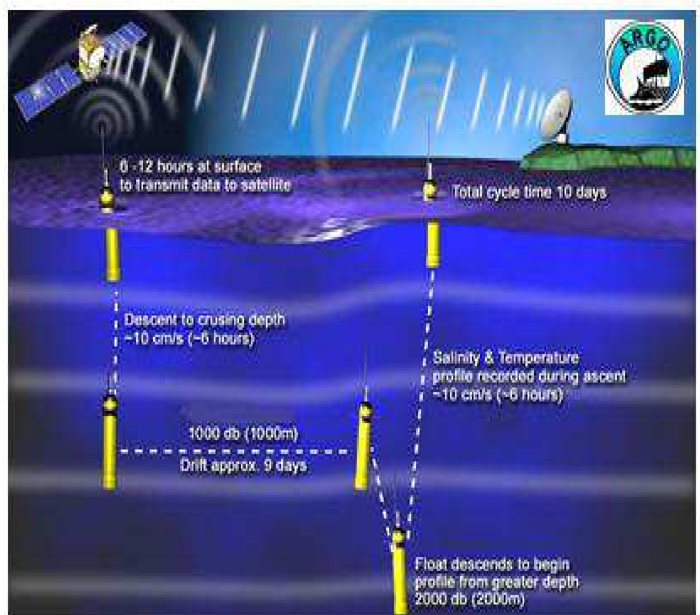
In view of the established properties of unique and dense solvability of the considered Problem Inv and data assimilation problem in each time interval, we can state that the system of all approximate solutions $\{\phi_j\}$ imparts the minimal value of whole cost functional, i.e., is the solution to the considered problem for the whole interval $(0, \bar{t}\,)$.

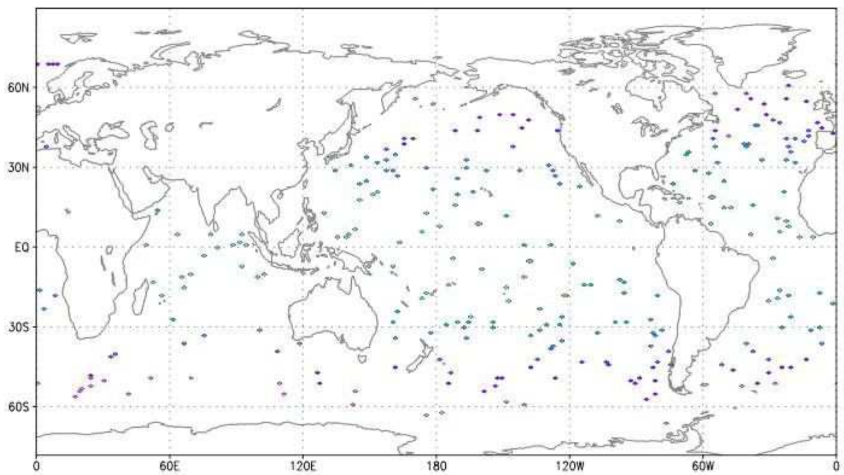
6. Information support of solving data assimilation problems

1. Data base (Lebedev S.A., 2005-2010, N.B.Zakharova 2011-2013)

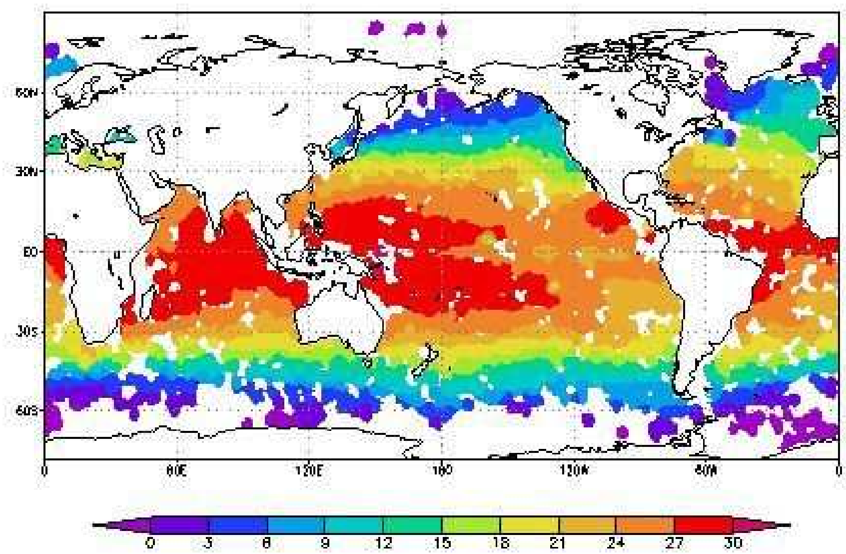
Data Base «World Ocean – INM RAS»





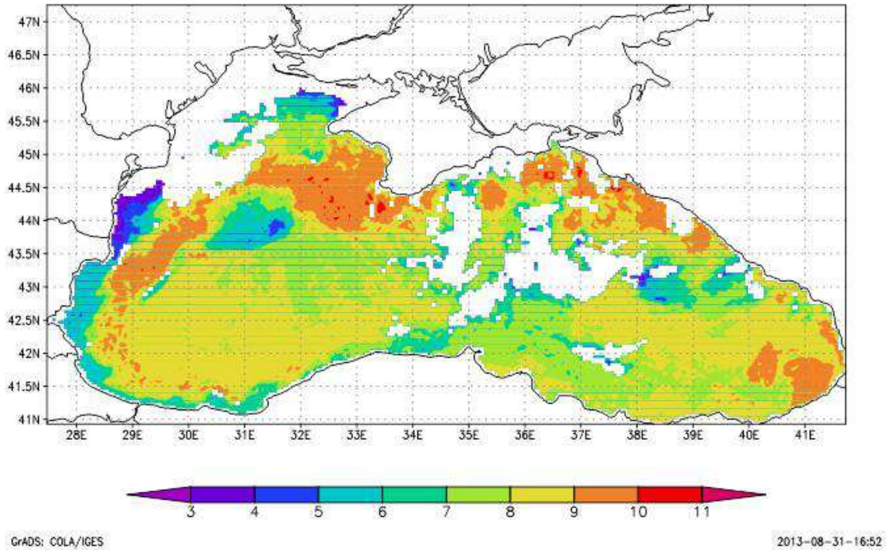


(a) Observation Data (ARGO)

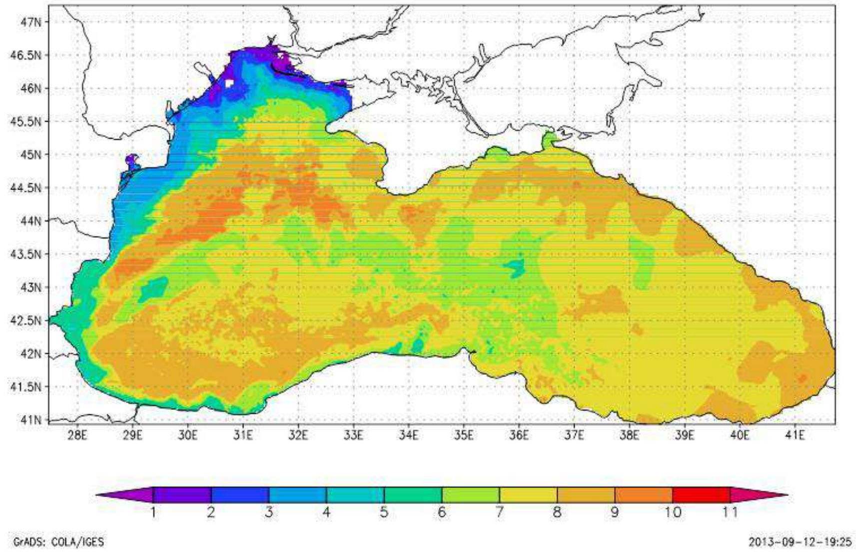


(b) SST after interpolation

6.2. New interpolation method. Black Sea(Agoshkov V.I., Zakharova N.B., Parmuzin E.I., 2013)



(c) Observation Data 13:58, 22th of January 2008



(d) SST after interpolation. The radius of the "neighbourhood" of the nodal point is 0.2 degrees, $\Delta k = 12$ hours

7. Numerical solving some variational data assimilation problems in the Black Sea hydrothermodynamics model.

7.1. Mathematical model of the Black Sea hydrothermodynamics

Suppositions

- The Earth is the GEOID (Listing, 1872)
- All terms of the order $\varepsilon \equiv (a-b)/a$ in the mathematical models are omitted
- The surface $z = 0$ is the surface of the "Normal Earth" ellipsoid
- The mathematical model was write down using "formally" spherical coordinates and the "spherical approximations" is taking
- The Boussinesq and hydrostatic approximations are introduced
- The viscosity forces are described by elliptic operators of 2-nd and 4-th orders applying scalar functions (Zalesny V.B., Diansky N.A., Gusev A.V.)

Mathematical model

$$\frac{d\vec{u}}{dt} + \begin{bmatrix} 0 & -f \\ f & 0 \end{bmatrix} \vec{u} - g \cdot grad\xi + A_u\vec{u} + (A_k)^2\vec{u} = \vec{f} - \frac{1}{\rho_0} gradP_a - \frac{g}{\rho_0} grad\int_0^z \rho_1(T,S)dz',$$

$$\frac{\partial \xi}{\partial t} - m\frac{\partial}{\partial x}(\int_0^H \Theta(z)udz) - m\frac{\partial}{\partial y}(\int_0^H \Theta(z)\frac{n}{m}vdz) = f_3,$$

$$\frac{dT}{dt} + A_T T = f_T,\ \frac{dS}{dt} + A_S S = f_S,$$

where $\vec{u} = (u, v)$,
$\Theta(z) \equiv \frac{r(z)}{R}$, $\quad r = R - z$, $\quad 0 < z < H, x \equiv \lambda, y \equiv \theta, n \equiv 1/r,\ m \equiv 1/(r\cos\theta)$. The functions $f_3, \xi_0 \equiv \xi$ at $t = 0$ will be "additional unknowns which must be calculated too.

Mathematical model

Here:

$$\vec{f} = \nabla\left[\widetilde{\Omega}_{np} - \beta\gamma_0\zeta_{0,np}\right] + \nabla\Psi,$$

where

$$\zeta_{0,np} \equiv A_{M_0}\zeta_{M_0} + A_{S_0}\zeta_{S_0} = A_0(1 - 3\sin^2\theta)/2, A_0 = 73,87[sm],$$

$$\widetilde{\Omega}_{np} \cong \gamma_0(A_{M_0}\zeta_{M_0} + A_{S_0}\zeta_{S_0}) + \sum_{j=1}^{4}\gamma_j C_j \cos(\sigma_j t + s_j\lambda + q_j),$$

or

$\widetilde{\Omega}_{np} = \gamma_0\Omega_{np}$

$j = O_1, K_1, M_2, S_2,$

$\Psi(\lambda, \Theta, t)$ - "additional potential" of "self-attraction" forces

$\beta = 0$, if ζ is calculated from the geoid surface,

$\beta = 1$, if ζ is calculated from the mean see level.

The function $\Psi(\lambda, \theta, t)$ can be unknown (then the function $G \equiv \widetilde{\Omega}_{np} - \beta\gamma_0\zeta_{0,np} + \Psi$ is unknown also).

Boundary conditions on the "sea surface" $\Gamma_S \equiv \Omega$ at $z = 0$:

$$
\begin{cases}
\left(\int_0^H \Theta \vec{u} dz \right) \vec{n} + \beta_0 m_{op} \sqrt{gH}\, \xi = m_{op} \sqrt{gH}\, d_s \text{ on } \partial\Omega, \\
U_n^{(-)} u - \nu \dfrac{\partial u}{\partial z} - k_{33} \dfrac{\partial}{\partial z} A_k u = \tau_x^{(a)} \Big/ \rho_0, \; U_n^{(-)} v - \nu \dfrac{\partial v}{\partial z} - k_{33} \dfrac{\partial}{\partial z} A_k v = \tau_y^{(a)} \Big/ \rho_0, \\
A_k u = 0, \quad A_k v = 0, \\
U_n^{(-)} T - \nu_T \dfrac{\partial T}{\partial z} + \gamma_T (T - T_a) = Q_T + U_n^{(-)} d_T, \\
U_n^{(-)} S - \nu_S \dfrac{\partial S}{\partial z} + \gamma_S (S - S_a) = Q_S + U_n^{(-)} d_S,
\end{cases}
$$

where
$U_n = \vec{U} \cdot \vec{N}, \vec{U} = (u, v, w) \equiv (\vec{u}, w), \vec{N} = (n_1, n_2, n_3) \equiv (\vec{n}, n_3), U_n^{(-)} = (|U_n| - U_n)/2.$
The boundary function d_T, d_S or Q_T, Q_S can be unknown also.

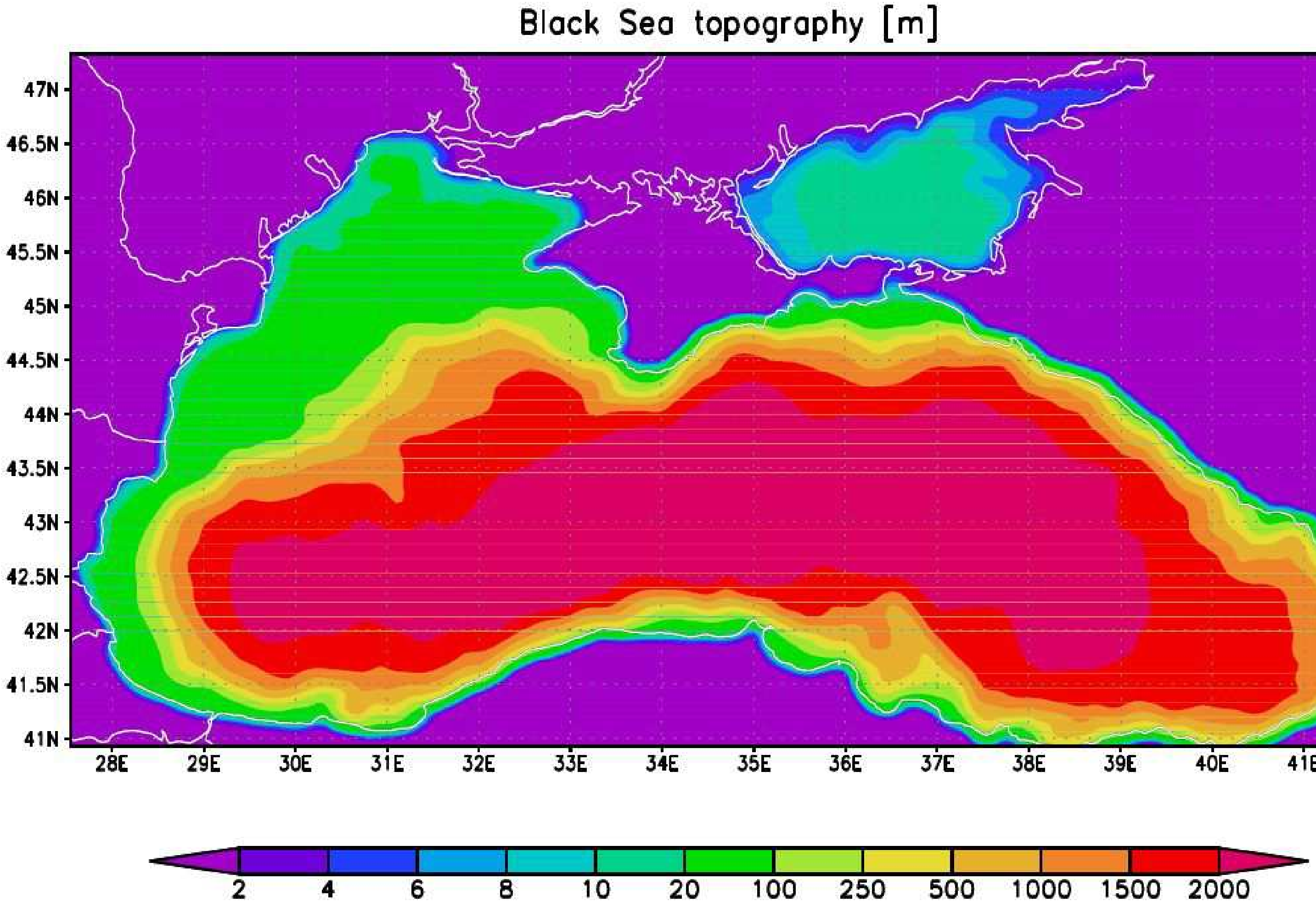
With the function $\phi = (u, v, \xi, T, S)$ known, we calculate

$$w(x,y,z,t) = \frac{1}{r}(m\frac{\partial}{\partial x}(\int_z^H rudz') + m\frac{\partial}{\partial y}(\frac{n}{m}\int_z^H rvdz')), (x,y,t) \in \Omega \times (0, \bar{t}),$$

$$P(x,y,z,t) = P_a(x,y,t) + \rho_0 g(z-\xi) + \int_0^z g\rho_1(T,S)dz'.$$

Note, that for $U_n \equiv \underline{U} \cdot \underline{N}$ (here $U = (u, v, w)$) we always have

$$U_n = 0 \text{ on } \Gamma_{c,w} \cup \Gamma_H.$$

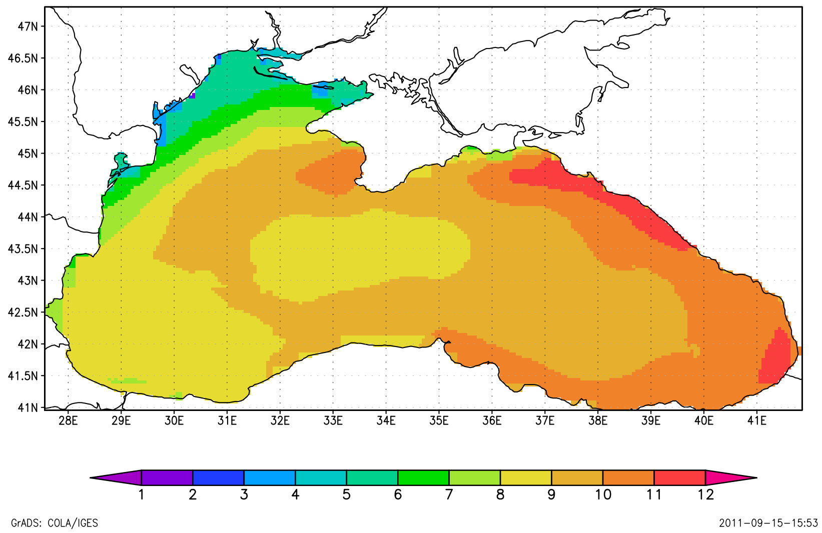

Black Sea topography [m]
47N
46.5N
46N
45.5N
45N
44.5N
44N
43.5N
43N
42.5N
42N
41.5N
41N
28E 29E 30E 31E 32E 33E 34E 35E 36E 37E 38E 39E 40E 41E
2 4 6 8 10 20 100 250 500 1000 1500 2000

- Agoshkov V.I. and Zalesny V.B. Variation data assimilation technique in mathematical modeling of ocean dynamics. – Pure Appl. Geophys., 2011 Springer-Basel AG
- Agoshkov V.I., Parmuzin E.I., Shutyaev V.P. A numerical algorithm of variational data assimilation for reconstruction of salinity fluxes on the ocean surface, Russ. J. Numer. Anal. Math. Modelling, Vol. 23, No. 2, 2008, pp. 135-161;
- V. I. Agoshkov, S. A. Lebedev, and E. I. Parmuzin Numerical Solution to the Problem of Variational Assimilation of Operational Observational Data on the Ocean Surface Temperature, Izvestiya, Atmospheric and Oceanic Physics, 2009, Vol. 45, No. 1, pp. 69–101;
- V. I. Agoshkov, N. B. Zakharova, and E. I. Parmuzin The study and numerical solution of the inverse problem of heat flows in the ocean dynamics model based on ARGO buoys data. Russ. J. Numer. Anal. Math. Modelling, 2011, V. 26, No. 3, pp. 231-261

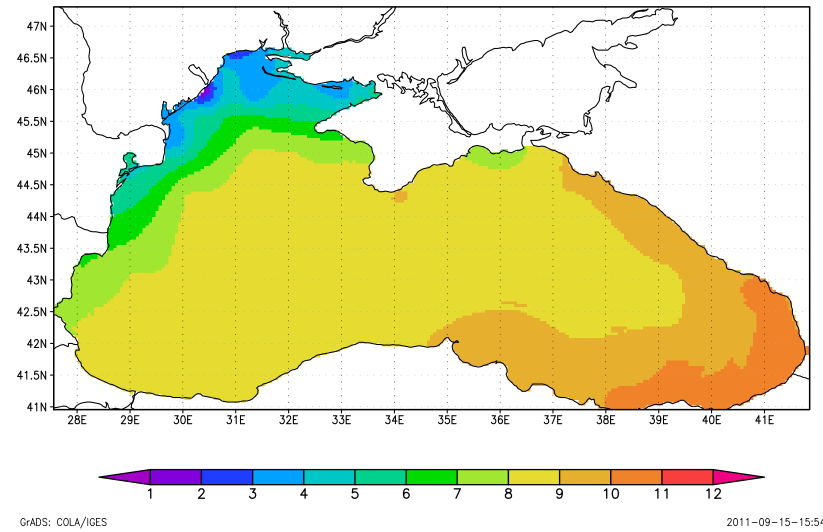
7.2 Numerical experiments

The object of simulation is the Black Sea. We can describe the parameters of the area studied and its geographical coordinates are: the grid 286x159x27 (latitude×longitude×depth). The grid steps with respect to x and y are constant and equal 0.05 and 0.04 degrees, respectively. The time step is equal to $\Delta t = 5$ minutes.
The data of SST, which was obtained from Geophysical Center of RAS (Lebedev S.A.), were used for the construction of the function T_{obs} at certain time steps at some points of Black Sea basin.
The mean flux for January $Q^{(0)}$ was taken from the database of NCEP (National Centers for Environmental Prediction).
The observation data assimilation module to assimilate T_{obs} was included into the thermohydrodynamics model of the Black Sea. The time period taken in experiments is 5 days (start from January 2008).

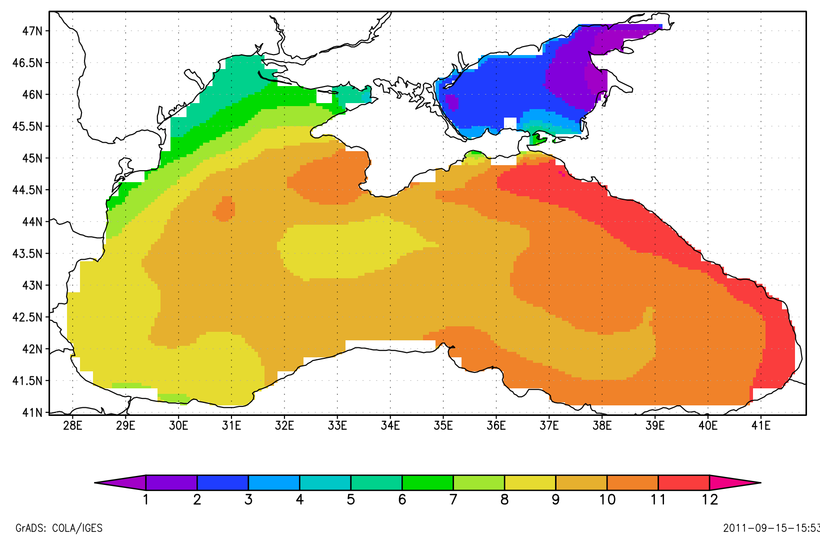
7.2.1 Numerical experiments: the Black Sea circulation model, SST assimilation, calculations of the flux Q at $z = 0$



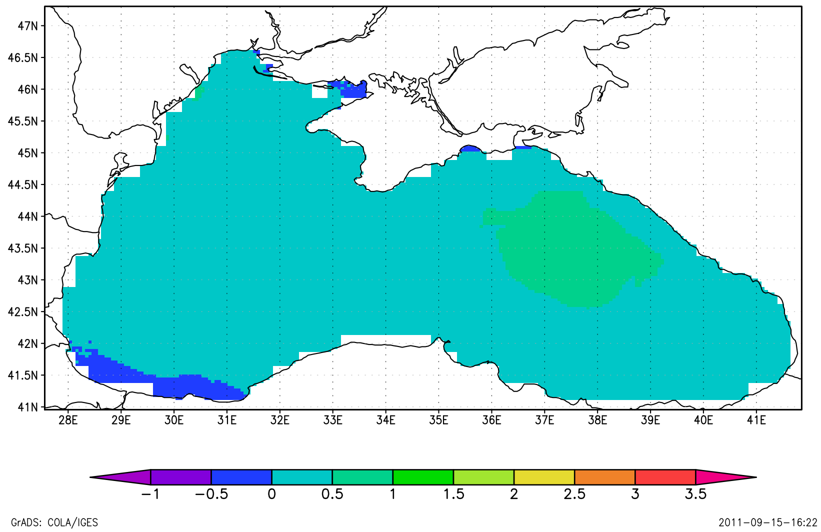
(a) Calculation using data assimilation



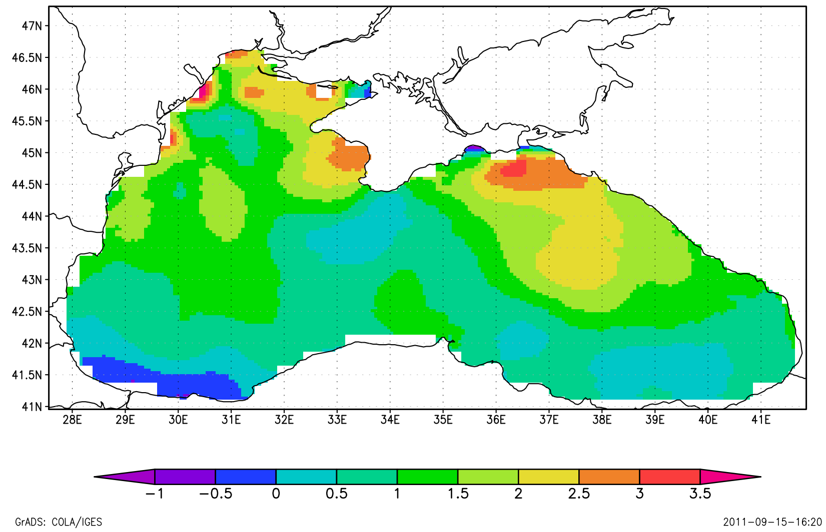
(b) Calculation by the model



(c) Observation of SST

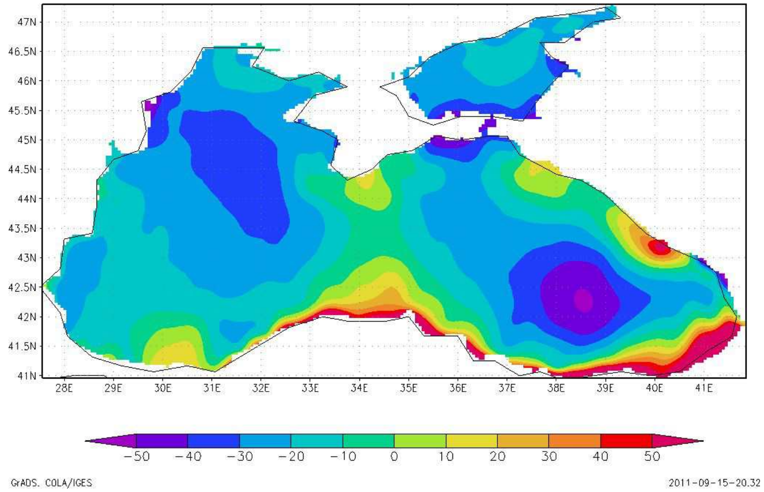


(a) Deviation assimilated SST from observations

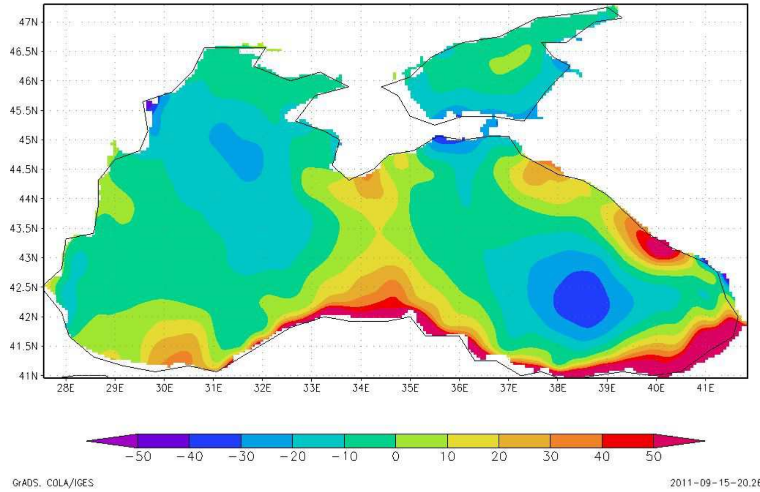


(b) Deviation model SST from observations

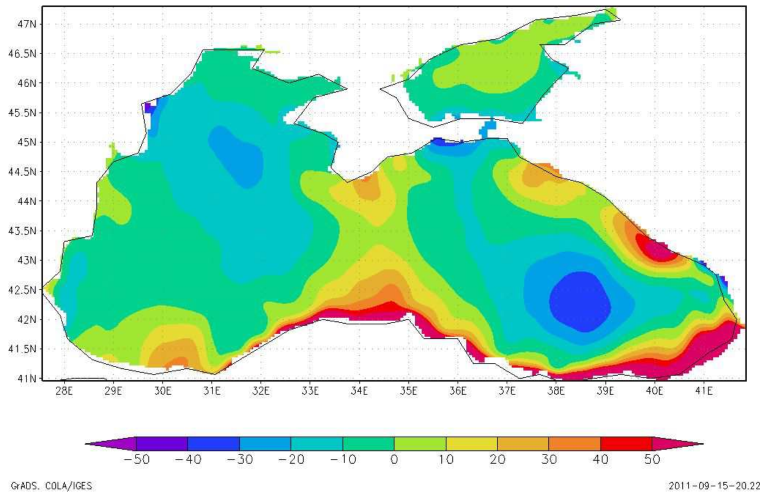
7.2.2 Numerical experiments: The Black Sea and Azov Sea circulation models, assimilation of mean sea level, calculations of the "self-attraction" forces, the tide potential is of complete form



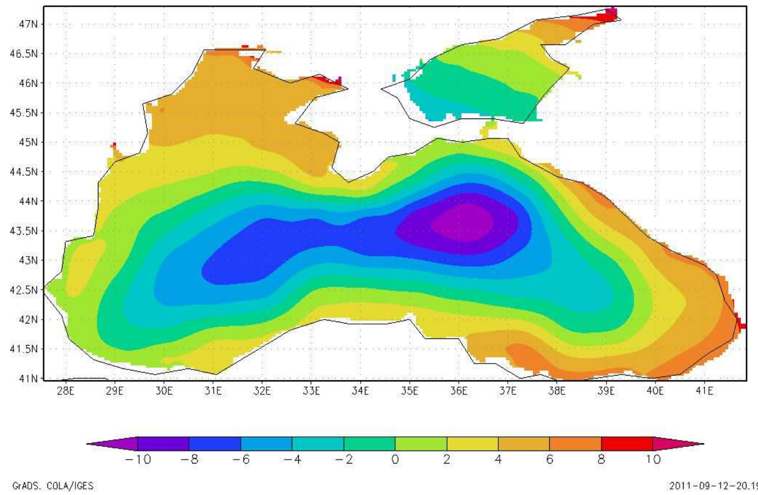
(a) Mean sea level (MSL)



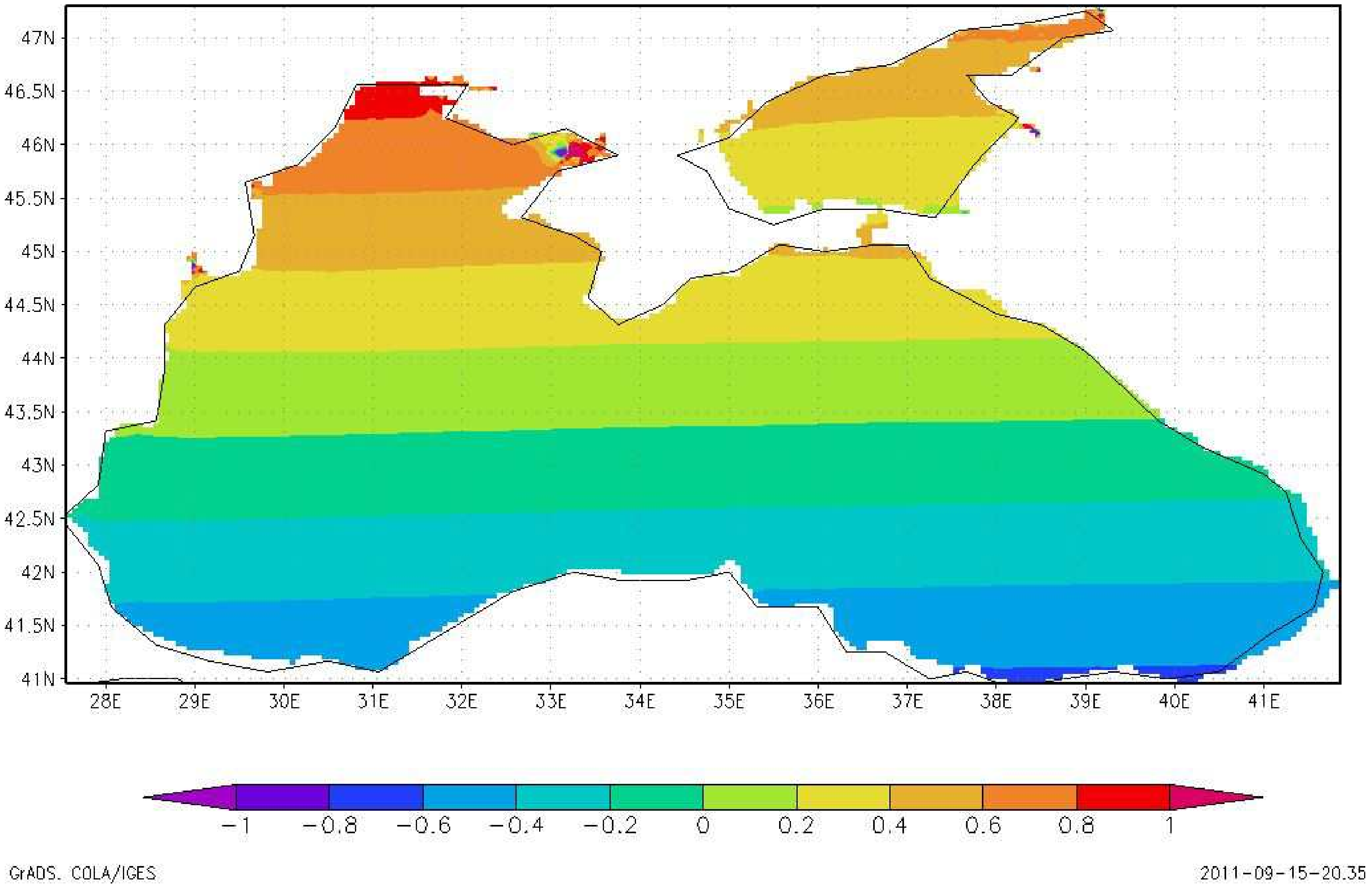
(b) MSL + const



(c) MSL (with the complete potential and assimilation)

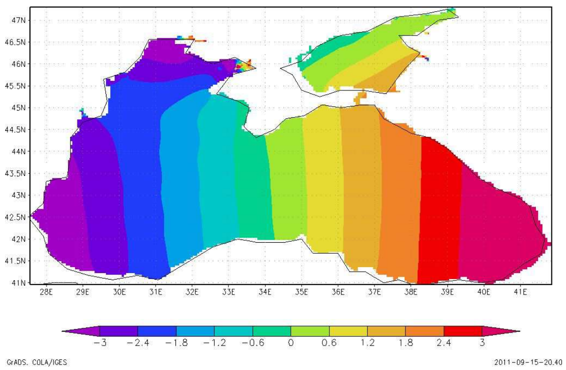


(d) MSL by the model (without tidal forces and assimilation)

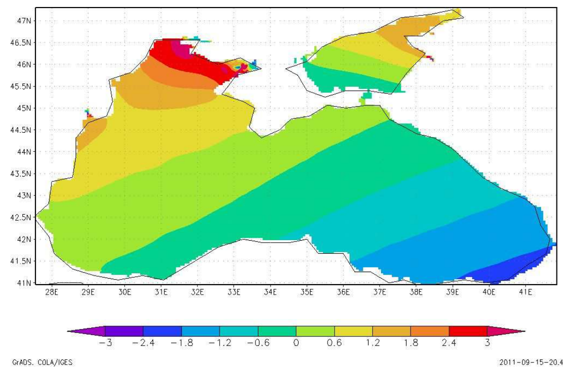


(a) The influence of the constant harmonics of the tide potential

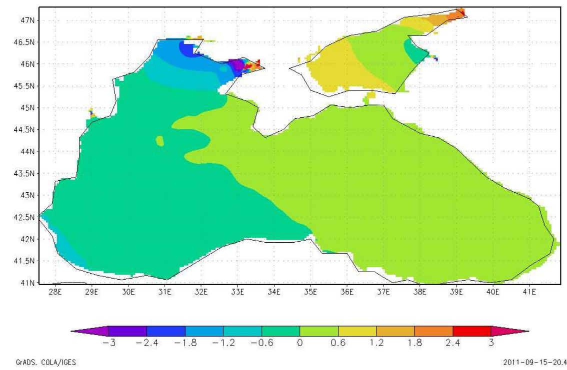
Tides in 5th January 2008



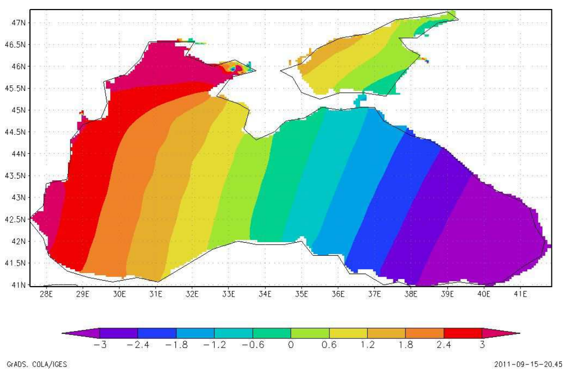
(a) tides0h



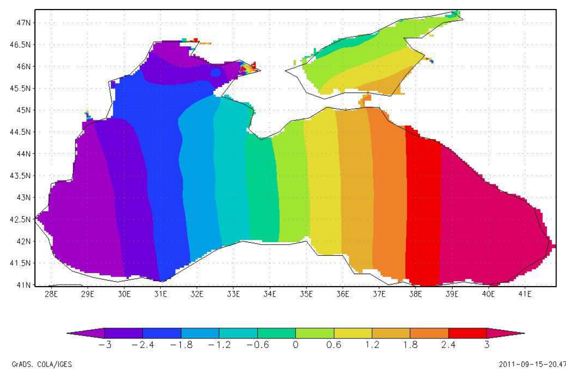
(b) tides6h



(c) tides12h



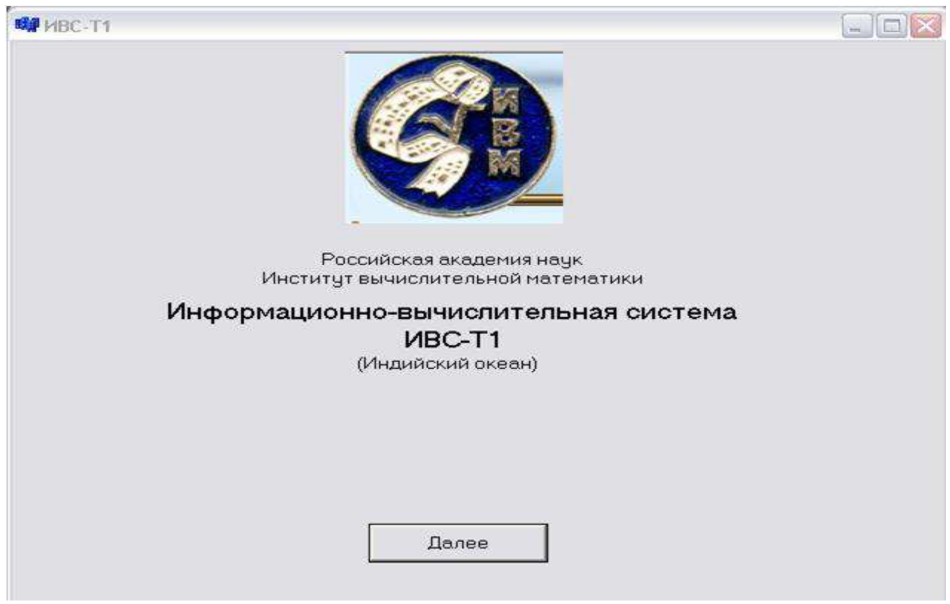
(d) tides18h



(e) tides24h

Conclusion

- The results obtained in INM RAS are the theoretical background for the construction of Informational-computational systems for variational data assimilation into the ocean circulation model and into the Black Sea model.



- Next step of the studying:
  - the improving the mathematical model (liquid boundaries conditions, input data and oth.)
  - algorithm of the coupling of the Black Sea and Azov Sea mathematical models
  - using the empirical relations in data assimilations procedures


Thank You!