NUMERICAL SOLVING OF VARIATIONAL DATA ASSIMILATION PROBLEMS IN THE BLACK SEA HYDROTHERMODYNAMICS MODEL USING REAL-TIME DATA

E.I. Parmuzin

Institute of Numerical Mathematics, RAS

1. Mathematical formulation of the problem

$$\frac{d\vec{u}}{dt} + \begin{bmatrix} 0 & -f \\ f & 0 \end{bmatrix} \vec{u} - g \cdot grad\xi + A_u\vec{u} + (A_k)^2\vec{u} = \vec{f} - \frac{1}{\rho_0} gradP_a - \frac{g}{\rho_0} grad \int\limits_0^z \rho_1(T,S)dz',$$

$$\frac{\partial \xi}{\partial t} - m\frac{\partial}{\partial x}(\int\limits_0^H \Theta(z)udz) - m\frac{\partial}{\partial y}(\int\limits_0^H \Theta(z)\frac{n}{m}vdz) = f_3,$$

$$\frac{dT}{dt} + A_T T = f_T, \ \frac{dS}{dt} + A_S S = f_S,$$

where

$$\bar{f} = g \cdot gradG, \ \Theta(z) \equiv \frac{r^2(z)}{R^2}, \quad r = R - z, \quad 0 < z < H.$$

Boundary conditions on the surface

$$\begin{cases} \left(\int\limits_0^H \Theta \vec{u} dz\right) \vec{n} + \beta_0 m_{op} \sqrt{gH}\, \xi = m_{op} \sqrt{gH}\, d_s \text{ on } \partial\Omega, \\ U_n^{(-)} u - \nu \dfrac{\partial u}{\partial z} - k_{33} \dfrac{\partial}{\partial z} A_k u = \tau_x^{(a)} \Big/ \rho_0, \; U_n^{(-)} v - \nu \dfrac{\partial v}{\partial z} - k_{33} \dfrac{\partial}{\partial z} A_k v = \tau_y^{(a)} \Big/ \rho_0, \\ A_k u = 0, \quad A_k v = 0, \\ U_n^{(-)} T - \nu_T \dfrac{\partial T}{\partial z} + \gamma_T (T - T_a) = Q_T + U_n^{(-)} d_T, \\ U_n^{(-)} S - \nu_S \dfrac{\partial S}{\partial z} + \gamma_S (S - S_a) = Q_S + U_n^{(-)} d_S. \end{cases}$$

The model approximation by splitting method

- General theory of splitting methods: G.I. Marchuk , N.N. Yanenko, A.A. Samarsky.
- Splitting method in data assimilation: Marchuk G.I., Zalesny V.B. (1993), M. Wenzel, V.B. Zalesny (1996), V.B. Zalesny (2005).
- Studies of inverse and assimilation problems for semidiscrete models in tidal dynamics problem: Agoshkov V.I. (2005-2007).
- Studies of class of data assimilation problems for ocean dynamics models obtained by splitting method: Agoshkov V.I. (2005, 2006).

Problem I

Step 1. We consider the system:

$$\begin{cases} T_t + (\bar{U}, \mathbf{Grad})T - \mathbf{Div}(\hat{a}_T \cdot \mathbf{Grad}\ T) = f_T \text{ in } D \times (t_{j-1}, t_j), \\ T = T_{j-1} \text{ for } t = t_{j-1} \text{ in } D, \\ \bar{U}_n^{(-)} T - \nu_T \dfrac{\partial T}{\partial z} + \gamma_T (T - T_a) = Q_T + \bar{U}_n^{(-)} d_T \text{ on } \Gamma_S \times (t_{j-1}, t_j), \\ \dfrac{\partial T}{\partial N_T} = 0 \text{ on } \Gamma_{w,c} \times (t_{j-1}, t_j), \\ \bar{U}_n^{(-)} T + \dfrac{\partial T}{\partial N_T} = \bar{U}_n^{(-)} d_T + Q_T \text{ on } \Gamma_{w,op} \times (t_{j-1}, t_j), \\ \dfrac{\partial T}{\partial N_T} = 0 \text{ on } \Gamma_H \times (t_{j-1}, t_j), \\ T_j \equiv T \text{ on } D \times (t_{j-1}, t_j). \end{cases}$$

Step 2.

$$
\begin{cases}
S_t + (\bar{U}, \mathbf{Grad})S - \mathbf{Div}(\hat{a}_S \cdot \mathbf{Grad}\ S) = f_S \text{ in } D \times (t_{j-1}, t_j),\\[2ex]
S = S_{j-1} \text{ at } t = t_{j-1} \text{ in } D,\\[2ex]
\bar{U}_n^{(-)} S - \nu_S \dfrac{\partial S}{\partial z} + \gamma_S (S - S_a) = Q_S + \bar{U}_n^{(-)} d_S \text{ on } \Gamma_S \times (t_{j-1}, t_j),\\
\dfrac{\partial S}{\partial N_S} = 0 \text{ on } \Gamma_{w,c} \times (t_{j-1}, t_j),\\
\bar{U}_n^{(-)} S + \dfrac{\partial S}{\partial N_S} = \bar{U}_n^{(-)} d_S + Q_S \text{ on } \Gamma_{w,op} \times (t_{j-1}, t_j),\\
\dfrac{\partial S}{\partial N_S} = 0 \text{ on } \Gamma_H \times (t_{j-1}, t_j),\\[2ex]
S_j \equiv S \text{ on } D \times (t_{j-1}, t_j).
\end{cases}
$$

Step 3.

$$\begin{cases} \underline{u}_t^{(1)} + \begin{bmatrix} 0 & -\ell \\ \ell & 0 \end{bmatrix} \underline{u}^{(1)} - g \cdot \mathbf{grad}\xi = g \cdot \mathbf{grad} G - \dfrac{1}{\rho_0}\mathbf{grad}\left(P_a + g\int\limits_0^z \rho_1(\bar{T}, \bar{S})dz'\right) \\ \text{ in } D \times (t_{j-1}, t_j), \\ \xi_t - \mathbf{div}\left(\int\limits_0^H \Theta \underline{u}^{(1)} dz\right) = f_3 \text{ in } \Omega \times (t_{j-1}, t_j), \\ \underline{u}^{(1)} = \underline{u}_{j-1},\ \xi = \xi_{j-1} \text{ at } t = t_{j-1}, \\ \left(\int\limits_0^H \Theta \underline{u}^{(1)} dz\right) \cdot n + \beta_0 m_{op}\sqrt{gH}\xi = m_{op}\sqrt{gH} d_s \text{ on } \partial\Omega \times (t_{j-1}, t_j), \\ \underline{u}_j^{(1)} \equiv \underline{u}^{(1)}(t_j) \text{ in } D \end{cases}$$

$$\begin{cases} \underline{u}_t^{(2)} + \begin{bmatrix} 0 & -f_1(\bar{u}) \\ f_1(\bar{u}) & 0 \end{bmatrix} \underline{u}^{(2)} = 0 \text{ in } D \times (t_{j-1}, t_j), \\ \underline{u}^{(2)} = \underline{u}_j^{(1)} \text{ при } t = t_{j-1} \text{ in } D, \\ \underline{u}_j^{(2)} \equiv \underline{u}^{(2)}(t_j) \text{ in } D, \end{cases}$$

Step 3. (continued)

$$
\begin{cases}
\underline{u}_t^{(3)} + (\bar{U}, \mathbf{Grad})\underline{u}^{(3)} - \mathbf{Div}(\hat{a}_u \cdot \mathbf{Grad}\ )\underline{u}^{(3)} + (A_k)^2 \underline{u}^{(3)} = 0 \text{ in } D \times (t_{j-1}, t_j), \\
\underline{u}^{(3)} = \underline{u}^{(2)} \text{ at } t = t_{j-1} \text{ in } D, \\
\bar{U}_n^{(-)} \underline{u}^{(3)} - \nu_u \dfrac{\partial \underline{u}^{(3)}}{\partial z} - k_{33} \dfrac{\partial}{\partial z}(A_k \underline{u}^{(3)}) = \dfrac{\underline{\tau}^{(a)}}{\rho_0}, A_k \underline{u}^{(3)} = 0 \text{ on } \Gamma_S \times (t_{j-1}, t_j), \\
U_n^{(3)} = 0, \dfrac{\partial U^{(3)}}{\partial N_u} \cdot \bar{\underline{\tau}}_w + \left(\dfrac{\partial}{\partial N_k} A_k \underline{u}^{(3)} \right) \cdot \underline{\tau}_w = 0, A_k \underline{u}^{(3)} = 0 \text{ on } \Gamma_{w,c} \times (t_{j-1}, t_j), \\
\bar{U}_n^{(-)} (\tilde{U}^{(3)} \cdot \underline{N}) + \dfrac{\partial \tilde{U}^{(3)}}{\partial N_u} \cdot \bar{N} + \left(\dfrac{\partial}{\partial N_k} A_k \underline{u}^{(3)} \right) \cdot \bar{N} = \bar{U}_n^{(-)} d, A_k \underline{u}^{(3)} = 0 \text{on } \Gamma_{w,op} \times (t_{j-1}, t \\
\bar{U}_n^{(-)} (\tilde{U}^{(3)} \cdot \bar{\tau}_w) + \dfrac{\partial \tilde{U}^{(3)}}{\partial N_u} \cdot \bar{\tau}_w + \left(\dfrac{\partial}{\partial N_k} A_k \underline{u}^{(3)} \right) \cdot \underline{\tau}_w = 0, A_k \underline{u}^{(3)} = 0 \text{ on } \Gamma_{w,op} \times (t_{j-1}, t_j), \\
\dfrac{\partial \underline{u}^{(3)}}{\partial N_u} = \dfrac{\tau^{(b)}}{\rho_0} \text{ on } \Gamma_H \times (t_{j-1}, t_j),
\end{cases}
$$

where

$$
\begin{aligned}
&\underline{u}^{(3)} = (u^{(3)}, v^{(3)}),\ \tau^{(a)} = (\tau_x^{(a)}, \tau_y^{(a)}), \\
&U^{(3)} = (u^{(3)}, w^{(3)}(u^{(3)}, v^{(3)})),\ \tilde{U}^{(3)} = (u^{(3)}, 0), \\
&\tau^{(b)} = (\tau_x^{(b)}, \tau_y^{(b)}).
\end{aligned}
$$

3. SST data assimilation problen

Let us assume, that the unique function which is obtained by observation data processing is the function T_{obs} on $\bar{\Omega} \equiv \Omega \cup \partial\Omega$ at $t \in (t_{j-1}, t_j)$, $j = 1, 2, \ldots, J$. Let by phisical meaning this function is an approximation to STT data on Ω, i.e to $T\Big|_{z=0}$. We permit that the function T_{obs} is known only on the part of $\Omega \times (0, \bar{t})$ and we define a support of this function as m_0. Beyond of this area we suppose function T_{obs} is trivial. Let the function of full flux Q is an additional unknown function ("control") and we consider the cost-function in the form:

$$J_\alpha \equiv J_\alpha(Q, \phi) = \frac{1}{2} \int\limits_0^{\bar{t}} \int\limits_\Omega \alpha |Q - Q^{(0)}|^2 d\Omega dt + J_0(\phi),$$

$$J_0(\phi) = \frac{1}{2} \int\limits_0^{\bar{t}} \int\limits_\Omega m_0 |T - T_{obs}|^2 d\Omega dt.$$

Here $\alpha \equiv \alpha(\lambda, \theta, t)$ is a regularization function(is it possible, that $\alpha(\lambda, \theta, t) =$const$\geq 0$) and it may be a dimensional quantity; $Q^{(0)} \equiv Q^{(0)}(\lambda, \theta, t)$ is a given function.

We can formulate the data assimilation problem : *find the solution* ϕ *of the Problem* I *and function* Q*, such that, the functional* J_α *is minimal on the set of the solutions.*

The optimality system obtained consist of successive solving the variational assimilation problem on intervals $t \in (t_{j-1}, t_j)$, $j = 1, 2, \dots, J$ (Agoshkov V.I., 2006). The method can be discribed as follows:

STEP 1. We solve system of equations, which arise from minimization of the functional J_α on the set of the solution of the equations.

$$\begin{cases} (T_1)_t + L_1 T_1 = \mathcal{F}_1, \quad t \in (t_{j-1}, t_j), \\ T_1 = T_{j-1} \quad \text{at} \quad t = t_{j-1} \end{cases} \quad \begin{cases} (T_2)_t + L_2 T_2 = \mathcal{F}_2 + BQ_T, \quad t \in (t_{j-1}, t_j), \\ T_2(t_{j-1}) = T_1(t_j). \end{cases}$$

$$T_2(t_j) \equiv T_j \cong T \quad \text{at} \;\; t = t_j.$$

$$\begin{cases} (T_2^*)_t + L_2^* T_2^* = B^* m_0 (T - T_{obs}) \text{ in } D \times (t_0, t_1), \\ T_2^* = 0 \quad \text{for } t = t_1, \end{cases} \quad \begin{cases} (T_1^*)_t + L_1^* T_1^* = 0 \text{ in } D \times (t_0, t_1), \\ T_1^* = T_2^*(t_0) \quad \text{for } t = t_1 \end{cases}$$

$$\alpha(Q - Q^{(0)}) + T_2^* = 0 \quad \text{on} \;\; \Omega \times (t_0, t_1).$$

Functions T_2, $Q(t_1)$ are accepted as approximations to functions T, Q of the full solution for the Problem I at $t > t_1$, and $T_2(t_1) \cong T(t_1)$ is taken as an initial condition to solve the problem on the interval (t_1, t_2).

STEP 2. Solve problem for S:

$$S_t + (\bar{U}, \mathbf{Grad})S - \mathbf{Div}(\hat{a}_S \cdot \mathbf{Grad}\ S) = f_S \text{ in } D \times (t_0, t_1)$$

with corresponding boundary and initial conditions. After that the function S is accepted as an approximate solution, and the function $S(t_1)$ is taken as an initial condition for the problem for the interval (t_1, t_2).

STEP 3. Solve equations of the velocity module.

Iteration process

Given $Q^{(k)}$ one solve all subproblems from step 1, adjoint problem for this step and define new correction $Q^{(k+1)}$

$$Q^{(k+1)} = Q^{(k)} - \gamma_k(\alpha(Q^{(k)} - Q^{(0)}) + T_2^*) \quad \text{on} \;\; \Omega \times (t_0, t_1).$$

Parameters $\{\gamma_k\}$ can be calculated at $\alpha \approx +0$, by the property of dense solvability, as:

$$\gamma_k = \frac{1}{2} \; \frac{\int\limits_{t_0}^{t_1} \int\limits_{\Omega} (T - T_{obs})^2 \Big|_{\sigma=0} d\Omega dt}{\int\limits_{t_0}^{t_1} \int\limits_{\Omega} (T_2^*)^2 \Big|_{\sigma=0} d\Omega dt}.$$

4. Special features of data assimilation process

- Using the "Direct model" – Gusev A.V., Diansky N.A.
- Approximating all subproblems by finite difference methods in σ-coordinate system. (V.B. Zalesny, N.A. Diansky, A.V. Gusev).
- Using special formulae for calculation of SST and etc.
- Using some fast algorithms for solving data assimilation problems (Agoshkov V.I., Parmuzin E.I., Shutyaev V.P.).

5. Numerical experiments

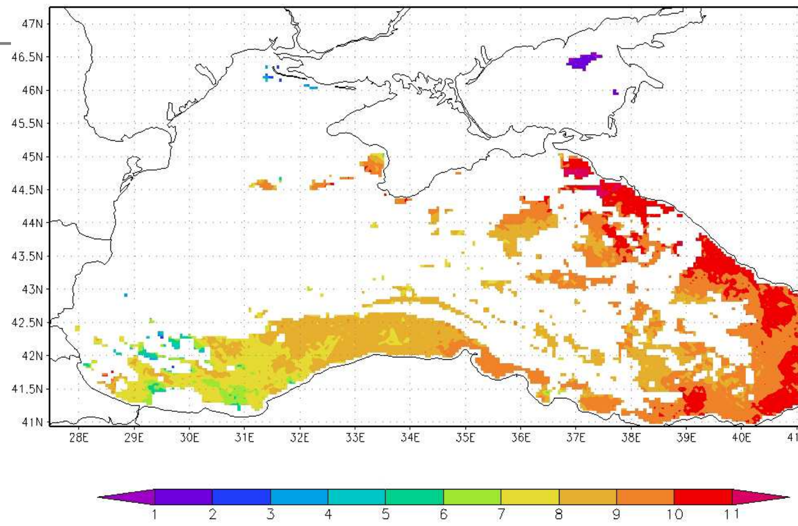
The object of simulation is the Black Sea. We can describe the parameters of the area studied and its geographical coordinates are: the grid 286x159x27 (latitude$\times$longitude$\times$depth). The grid steps with respect to x and y are constant and equal 0.05 and 0.04 degrees, respectively. The time step is equal to $\Delta t = 5$ minutes.

The data of SST, which was obtained from Geophysical Center of RAS (Lebedev S.A.), were used for the construction of the function T_{obs} at certain time steps at some points of Black Sea basin.

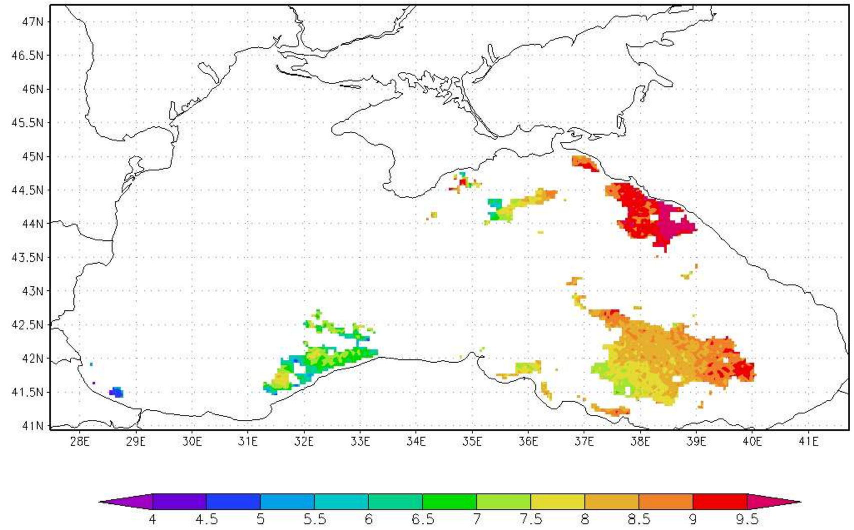
The mean flux for January $Q^{(0)}$ was taken from the database of NCEP (National Centers for Environmental Prediction).

The observation data assimilation module to assimilate T_{obs} was included into the thermohydrodynamics model of the Black Sea. The time period taken in experiments is 5 days (start from January 2008).

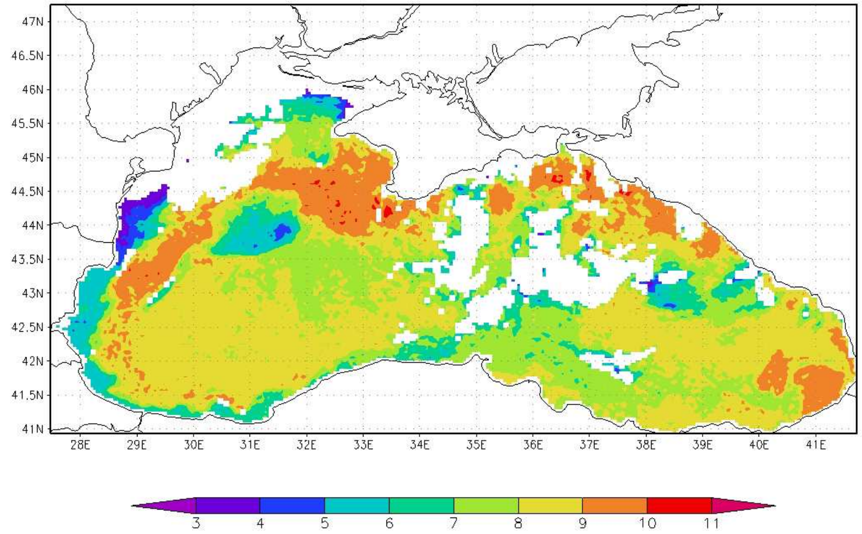
Observation data, January, 2008.



(a) 1st January 10:37

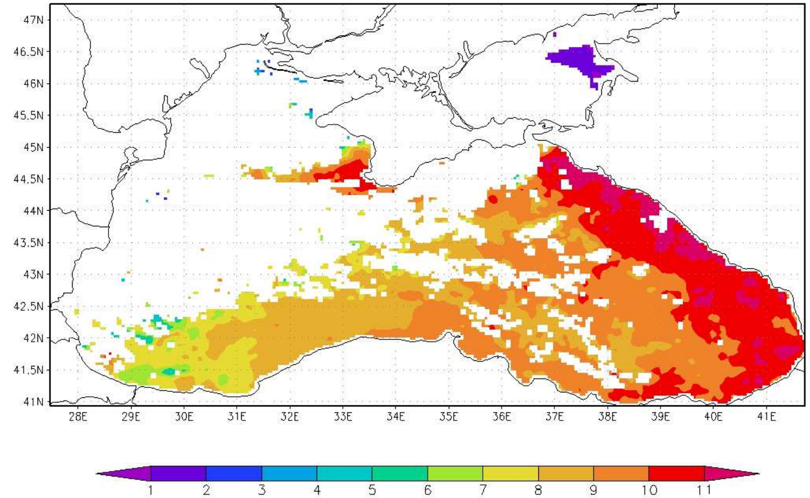


(b) 6th January 19:30

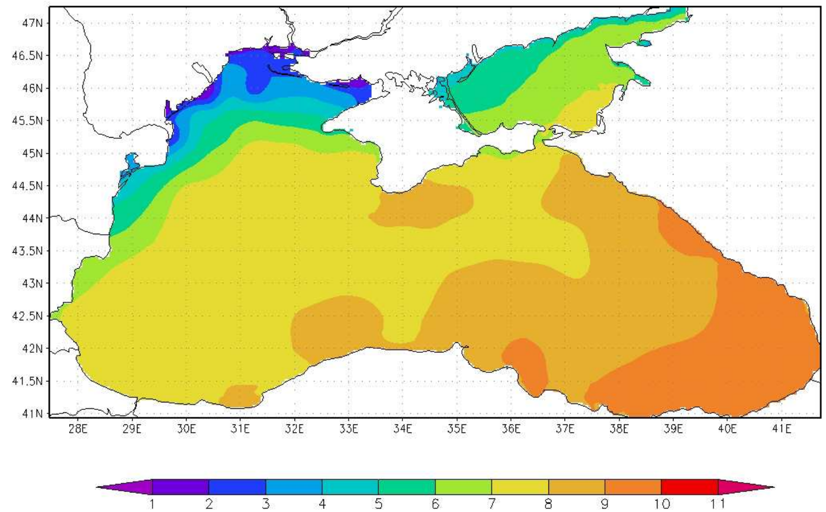


(c) 22nd January 13:58

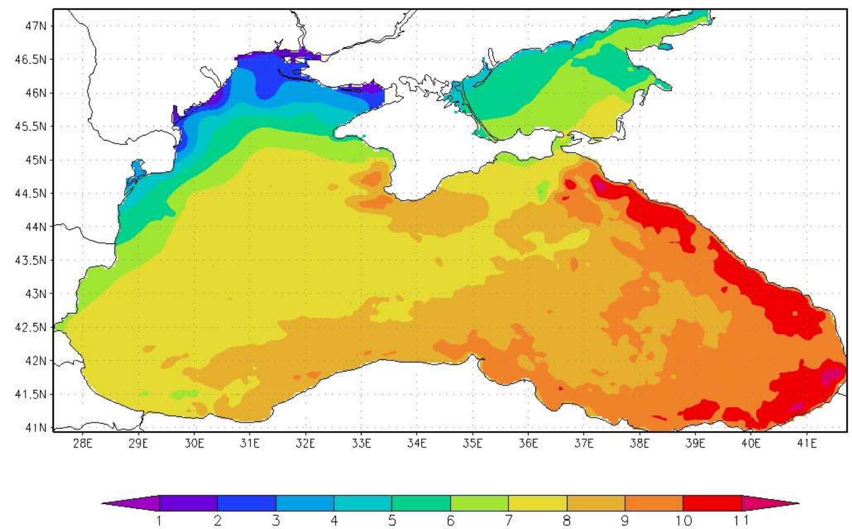
SST after 1 day of calculation, 1 January, 2008.



(a) Observation data (average SST)

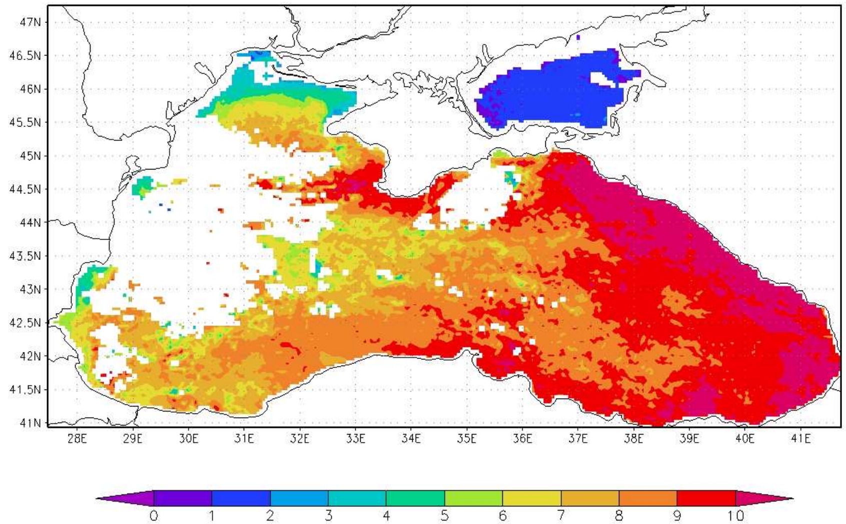


(b) Calculation without assimilation

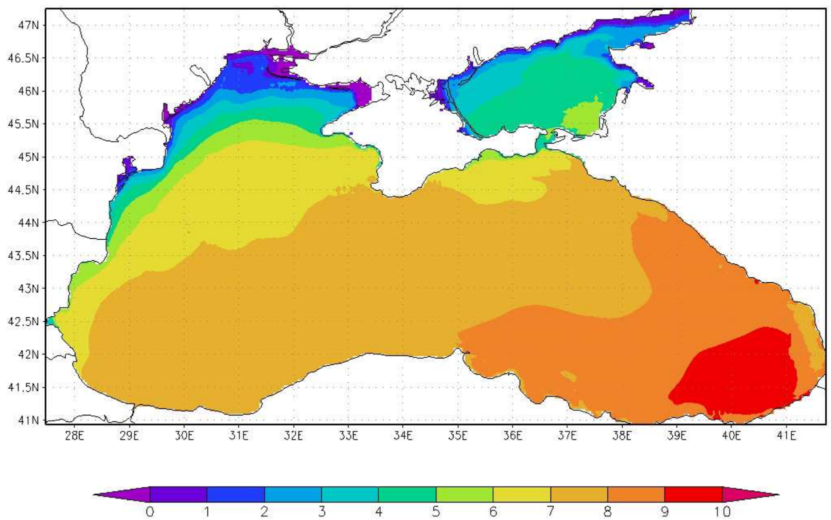


(c) Calculation with assimilation

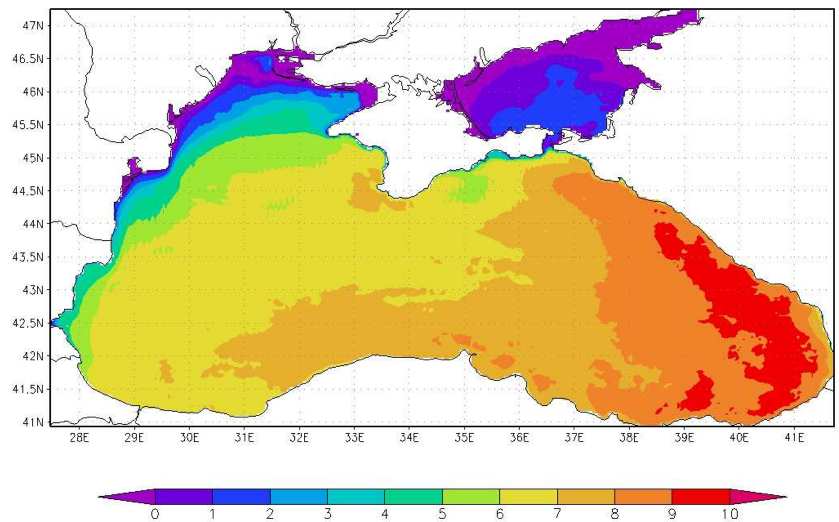
SST after 5 days of calculation, 1-5 January, 2008.



(a) Observation data (average SST)

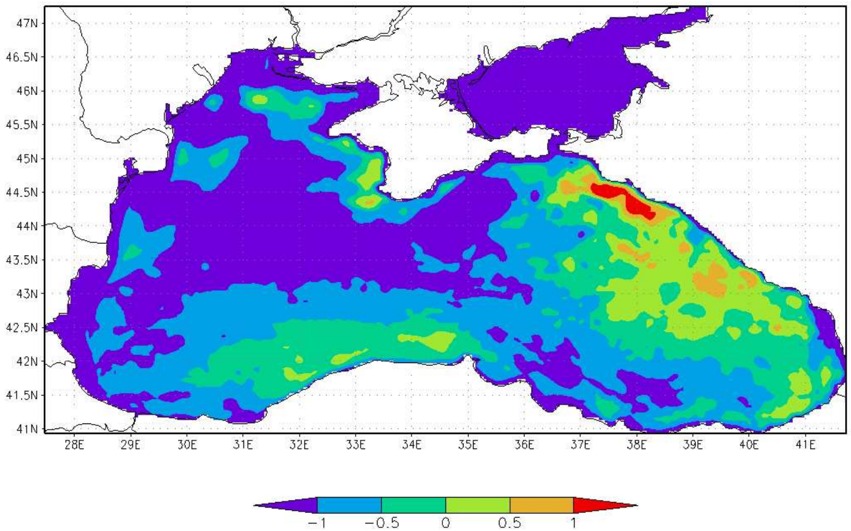


(b) Calculation without assimilation

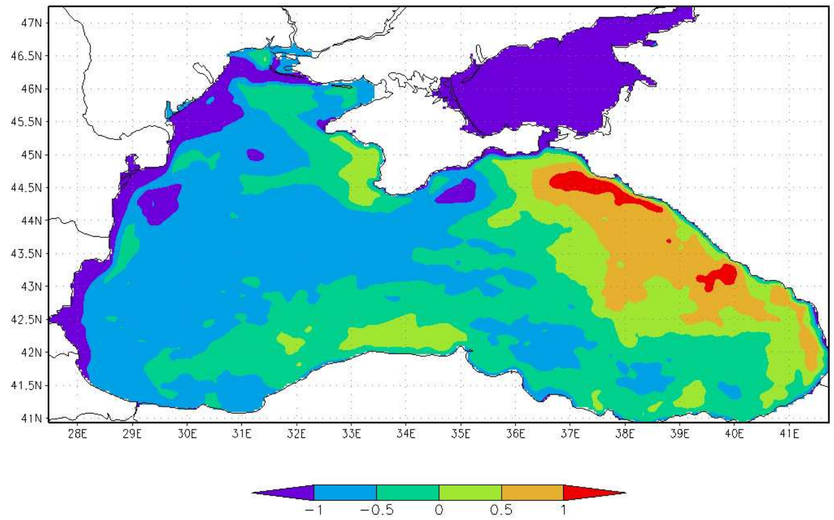


(c) Calculation with assimilation

Difference on SST



(a) $T_{assim} - T_{model}$ after 1 day



(b) $T_{assim} - T_{model}$ after 5 days

Conclusion

- The variational data assimilation problem of finding the flux on the sea surface using the observation of SST was formulated and studied.

- Algorithms of the numerical solution of data assimilation problem were developed and justified. The assimilation block was included into 3D hydrodynamics model developed in INM RAS.

- The numerical experiments show that assimilation of SST have a small influence to other components of the full solution, i.e. sea level function, velocity etc.

- The numerical experiments confirm the theoretical results and advisability of using the assimilation block in 3D model.

References

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Thank you