Sensitivity of the optimal solution in variational data assimilation for the sea thermodynamics model

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Thermodynamics equations

$$T_t + (U, Grad)T - Div(\hat{a}_T \cdot Grad\ T) = f_T \ in\ D \times (t_0, t_1),$$

$$T = T_0 \text{ for } t = t_0 \ on\ D,$$

$$-\nu_T \frac{\partial T}{\partial z} = Q \text{ on } \Gamma_S \times (t_0, t_1),$$

$$\frac{\partial T}{\partial N_T} = 0 \text{ on } \Gamma_{w,c} \times (t_0, t_1),$$

$$U_n^{(-)} T + \frac{\partial T}{\partial N_T} = U_n^{(-)} d_T + Q_T \text{ on } \Gamma_{w,op} \times (t_0, t_1),$$

$$\frac{\partial T}{\partial N_T} = 0 \text{ on } \Gamma_H \times (t_0, t_1).$$

Operator formulation of the forward problem

$$T_t + LT = F + BQ, \quad t \in (t_0, t_1),$$
$$T = T_0, \quad \text{for} \quad t = t_0,$$

in a week sense:

$$(T_t, \hat{T}) + (LT, \hat{T}) = F(\hat{T}) + (BQ, \hat{T}) \quad \forall \hat{T} \in W_2^1(D),$$

and *L, F, B,* are defined by

$$(LT, \hat{T}) \equiv \int_D (-TDiv(\bar{U}\hat{T})) + \int_{\Gamma_{w,op}} \bar{U}_n^{(+)} T\hat{T} d\Gamma + \int_D \hat{a}_T Grad(T) \cdot Grad(\hat{T}) dD,$$

$$F(\hat{T}) = \int_{\Gamma_{w,op}} (Q_T + \bar{U}_n^{(-)} d_T)\hat{T} dT + \int_D f_T \hat{T} dD, \quad (T_t, \hat{T}) = \int_D T_t \hat{T} dD, \quad (BQ, \hat{T}) = \int_\Omega Q\hat{T}|_{z=0} \, d\Omega.$$

SST data assimilation problem

$$\begin{cases} T_t + LT &= F + BQ \quad \text{in } D\times(t_0,t_1), \\ T &= T_0 \quad \text{for } t = t_0, \\ J(Q) &= \inf\limits_{Q} J(Q), \end{cases}$$

$$J(Q) = \frac{1}{2}\int_{t_0}^{t_1}\int_{\Omega} \alpha\,|Q - Q^{(0)}|^2\, d\Omega dt + \frac{1}{2}\int_{t_0}^{t_1}\int_{\Omega} m_0\,|T|_{z=0} - T_{\text{obs}}|^2\, d\Omega dt.$$

The optimality system:

$$T_t + LT = F + BQ \quad \text{in} \quad D \times (t_0, t_1),$$
$$T = T_0 \quad \text{for} \quad t = t_0,$$

$$-(T^*)_t + L^* T^* = B m_0 (B^* T - T_{\text{obs}}) \quad \text{in} \quad D \times (t_0, t_1),$$
$$T^* = 0 \quad \text{for} \quad t = t_1,$$

$$\alpha (Q - Q^{(0)}) + T^* = 0 \quad \text{on} \quad \Omega \times (t_0, t_1),$$

Input errors

$$Q^{(0)} = \overline{Q} + \xi_1, \; T_{obs} = \overline{T}\,|_{z=0} + \xi_2,$$

where $\delta T = T - \overline{T}, \delta Q = Q - \overline{Q}$ and

$$\overline{T}_t + L\overline{T} = F + B\overline{Q} \quad \text{in} \quad D \times (t_0, t_1),$$

$$\overline{T} = T_0 \quad \text{for} \quad t = t_0 .$$

System for the errors

$$\delta T_t + L\delta T = B\delta Q \quad \text{in} \quad D\times(t_0, t_1),$$
$$\delta T = 0 \quad \text{for} \quad t = t_0,$$

$$-(T^*)_t + L^* T^* = Bm_0(B^*\delta T - \xi_2) \quad \text{in} \quad D\times(t_0, t_1),$$
$$T^* = 0 \quad \text{for} \quad t = t_1,$$

$$\alpha(\delta Q - \xi_1) + T^* = 0 \quad \text{on} \quad \Omega\times(t_0, t_1).$$

Auxiliary minimization problem

$$\begin{cases} \delta T_t + L\delta T = B\delta Q & \text{in } D\times(t_0,t_1), \\ \delta T = 0 & \text{for } t = t_0, \\ S(\delta Q) = \inf_Q S(Q), \end{cases}$$

$$S(\delta Q) = \frac{1}{2}\int_{t_0}^{t_1}\int_{\Omega} \alpha\,|\,\delta Q - \xi_1|^2\, d\Omega dt + \frac{1}{2}\int_{t_0}^{t_1}\int_{\Omega} m_0\,|\,\delta T|_{z=0} - \xi_2|^2\, d\Omega dt.$$

Hessian of the functional

$$\psi_t + L\psi = Bv \quad \text{in} \quad D \times (t_0, t_1),$$

$$\psi = 0 \quad \text{for} \quad t = t_0,$$

$$-(\psi^*)_t + L^*\psi^* = Bm_0B^*\psi \quad \text{in} \quad D \times (t_0, t_1),$$

$$\psi^* = 0 \quad \text{for} \quad t = t_1,$$

$$Hv = \alpha v + \psi^* \quad \text{on} \quad \Omega \times (t_0, t_1).$$

Error control equation

$$H\,\delta Q = R_1\xi_1 + R_2\xi_2 .$$

$$R_1 = \alpha E,\; R_2\xi_2 = \theta^* |_{z=0},$$

$$-(\theta^*)_t + L^*\theta^* = Bm_0\xi_2 \quad \text{in} \quad D\times(t_0,t_1),$$

$$\theta^* = 0 \quad \text{for} \quad t = t_1 .$$

The optimal solution error:

$$\delta Q = T_1\xi_1 + T_2\xi_2, \qquad T_1 = H^{-1}R_1,\; T_2 = H^{-1}R_2 .$$

Sensitivity coefficients

$$r_1 = \sqrt{\left\| T_1^{\,*} T_1 \right\|}, \qquad r_2 = \sqrt{\left\| T_2^{\,*} T_2 \right\|}$$

$$r_1 = \frac{\alpha}{\mu_{\min}}, \qquad r_2 = \sqrt{\left\| (H - \alpha E) H^{-2} \right\|}.$$

Fundamental control functions

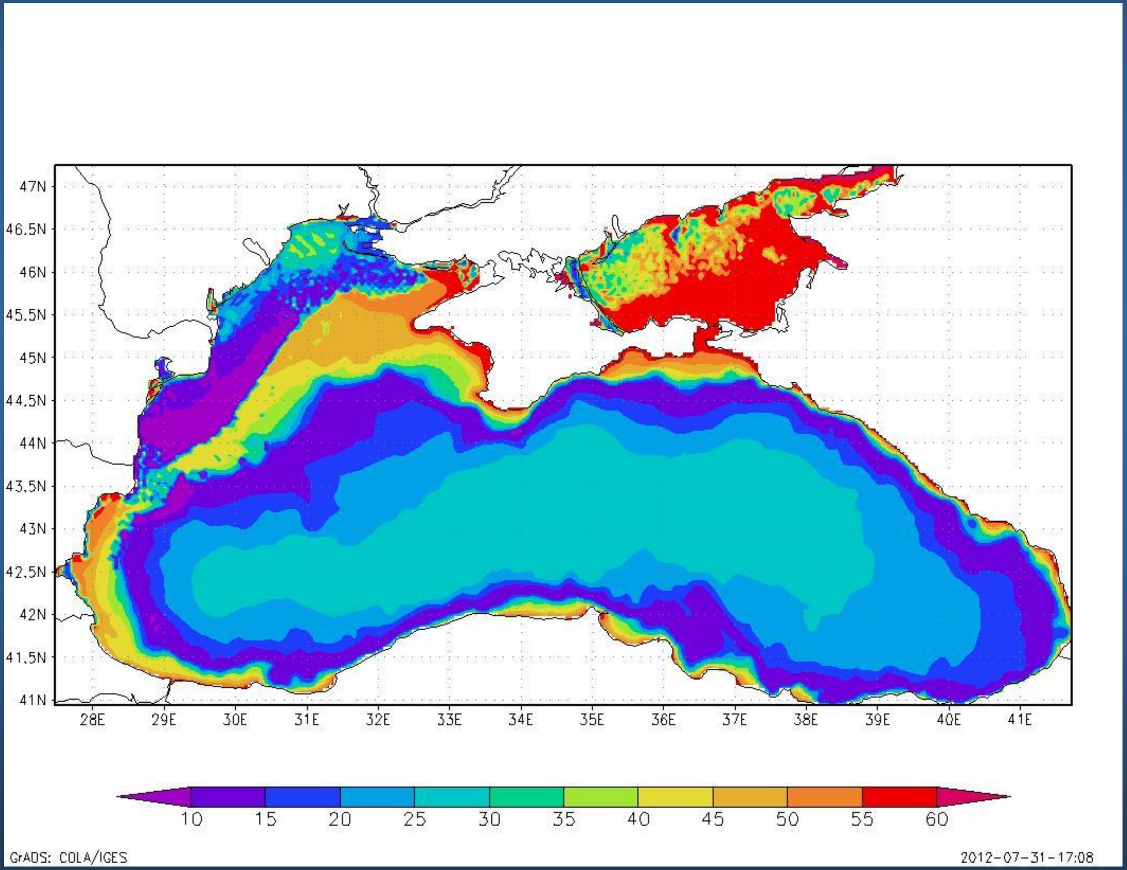
$$(\varphi_k)_t + L\varphi_k = Bv_k \quad \text{in} \;\; D\times(t_0,t_1),$$
$$\varphi_k = 0 \quad \text{for} \;\; t = t_0,$$

$$-(\varphi_k^*)_t + L^*\varphi_k^* = Bm_0B^*\varphi_k \quad \text{in} \;\; D\times(t_0,t_1),$$
$$\varphi_k^* = 0 \quad \text{for} \;\; t = t_1,$$
$$\alpha v_k + \varphi_k^* = \mu_k v_k \quad \text{on} \;\; \Omega\times(t_0,t_1).$$

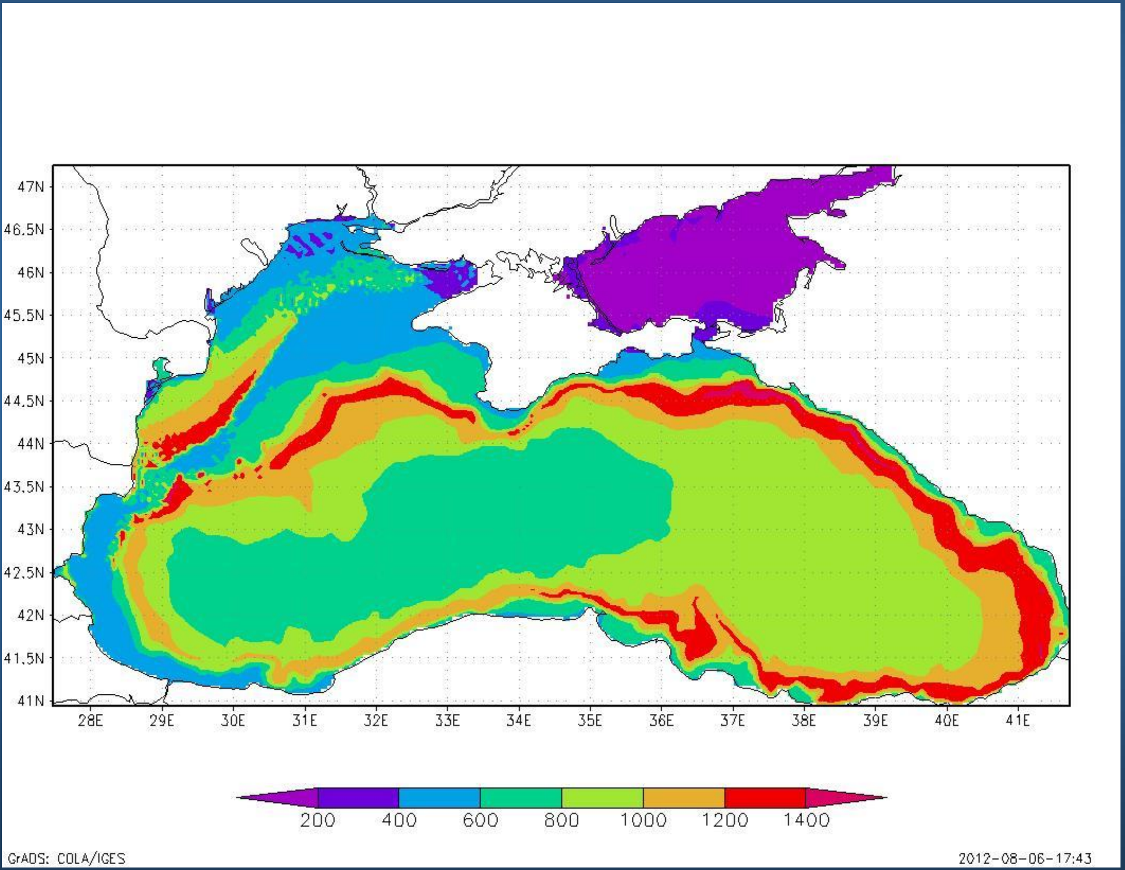
Singular vectors:

$$T_2^*T_2 w_k = \sigma_k^2 w_k, \quad w_k = \frac{1}{\sqrt{\mu_k - \alpha}}\varphi_k\,|_{z=0}, \qquad k = 1,2,\ldots \; \sigma_k^2 = \frac{\mu_k - \alpha}{\mu_k^2}.$$

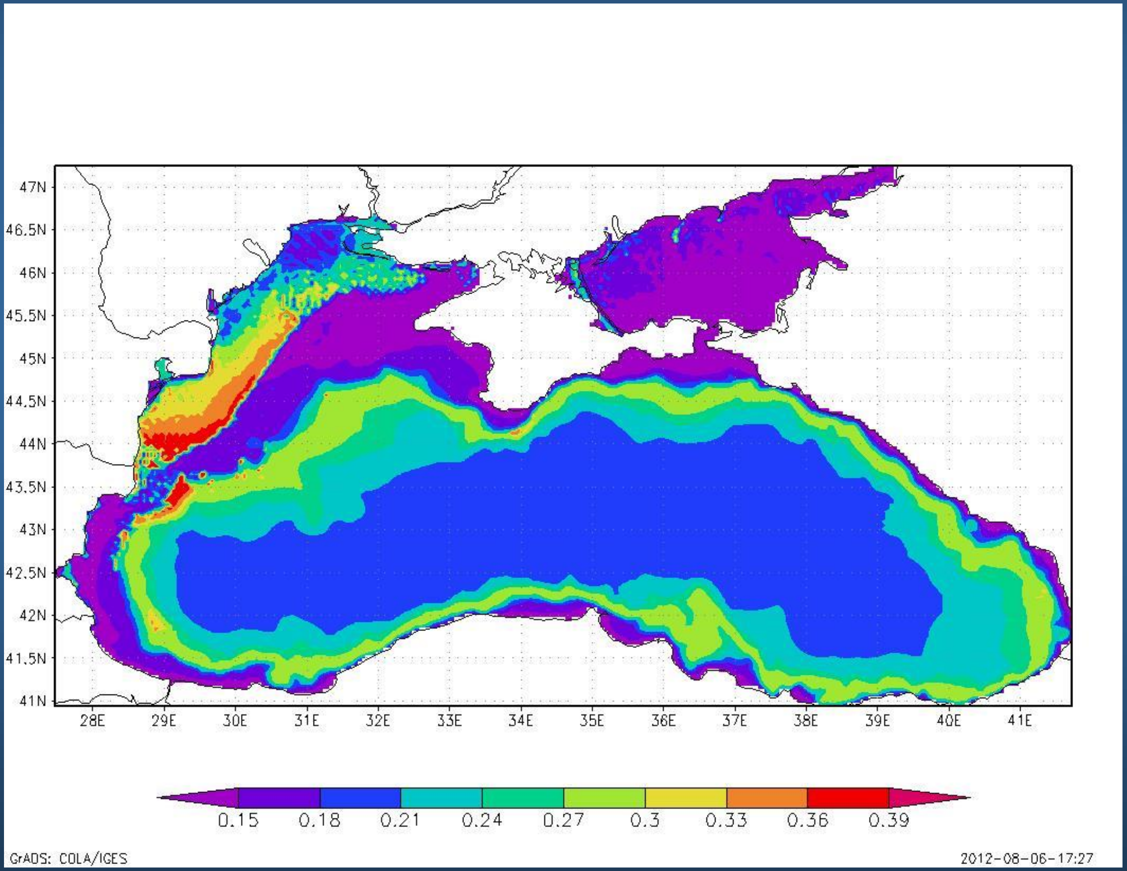
Numerical examples (Black Sea, t=20h50m)



Flux error (t=20h50m, ε=0.01)



Singular values of T_2



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Thank you for your attention!