Mathematical models and numerical methods of geophysical fluid dynamics - to the memory of G.I. Marchuk

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Dedicated to the memory of
Guri Ivanovich Marchuk – the last President of the USSR Academy of Sciences, the Founder of the Institute of Numerical Mathematics RAS
(1925 – 2013)

CONTENTS

- **Theorems: from conceptual models to global solutions of the PE ocean systems**
- **Numerical methods: from regular grids FDM to unstructured adaptive meshes, operator decomposition /splitting methods**
- **New problems: from forward to inverse/ optimal control problems**
- **Numerical experiments: from coarse to eddy-resolving modeling / high performance computing**

THEOREMS

$$
\begin{cases}
\dfrac{\partial u}{\partial t} - lv = -\dfrac{1}{\rho_0}\dfrac{\partial p}{\partial x} \\
\dfrac{\partial v}{\partial t} + lu = -\dfrac{1}{\rho_0}\dfrac{\partial p}{\partial y} \\
\dfrac{\partial p}{\partial z} = g\rho \\
\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0 \\
\dfrac{\partial \rho}{\partial t} + \Gamma(z)w = 0,
\end{cases}
$$

$$
\frac{\partial p}{\partial t} + g\rho_0 w = 0, \qquad z = 0 \qquad + b.c.
$$

- **Demidov, Marchuk (1966, атм)**
- **Bubnov, Kajikhov (1971), Kordzadze (1979, 1982), Bubnov (1978- 1984), Marchuk, Bubnov (1980), Bennet, Kloeden (1980, QG) et al.**
- **Sukhonosov (1981), Lions et al., (1992-1995), Dymnikov, Filatov (1997), Temam, Ziane (2004), Titi (2007), Kukavica, Ziane (2007)**
- **Kobelkov (2006, 2008)**
- **Agoshkov, Ipatova (2007, + assim., 2010)**

Solvability of PE ocean models (*Агошков, Ипатова, 2007*)

$$\begin{cases} \dfrac{d\vec{u}}{dt} + \begin{bmatrix} 0 & -f \\ f & 0 \end{bmatrix} \vec{u} - g\,\mathbf{grad}\,\xi + A_u \vec{u} + (A_k)^2 \vec{u} = \vec{f} - \dfrac{1}{\rho_0}\mathbf{grad} P_a - \\ \qquad -\dfrac{g}{\rho_0}\mathbf{grad}\displaystyle\int_0^z \rho_1(T,S)\,dz', \\ \dfrac{\partial \xi}{\partial t} - m\dfrac{\partial}{\partial x}\left(\displaystyle\int_0^H \Theta(z) u\,dz\right) - m\dfrac{\partial}{\partial y}\left(\displaystyle\int_0^H \Theta(z)\dfrac{n}{m} v\,dz\right) = f_3, \\ \dfrac{dT}{dt} + A_T T = f_T, \quad \dfrac{dS}{dt} + A_S S = f_S, \end{cases}$$

$$\rho_1(T,S) = \rho_0 \beta_T (T - T^{(0)}) + \rho_0 \beta_S (S - S^{(0)}) + \gamma \rho_0 \beta_{TS}(T,S) + f_P,$$

Some references

Drutsa A.V. Existence "in the large" of a solution to primitive equations in a domain with uneven bottom. RJNAMM, v. 24, no 6, 515-542, 2009.

Agoshkov V.I., Ipatova V.M. Convergence of solutions to the problem of data assimilation for a multilayer quasigeostrophic model of ocean dynamics. RJNAMM, v. 25, no 2, 105-115, 2010.

Ipatova V.M., Agoshkov V.I., Kobelkov G.M., Zalesny V.B. Theory of solvability of boundary value problems for ocean dynamics equations. RJNAMM, v. 25, no 6, 511-534, 2010

Numerical methods and sea/ocean dynamics models: from FDM to FEM / unstructured adaptive meshes

- **Methods: Sarkisyan (1954), Brayn (1969) – Box method / FVM**
- **Marchuk, Kuzin et al. (1969, 1975, 1983) – FEM + FDM**
- **Oganesyan et al. (1979-2006) - FEM, multigrid**
-
- **Models: MOM (Brayn et al., 1969-2013)**
- **CC SBAS (Марчук, Kordzadze et al., 1975, 1984)**
- **OPA/NEMO (Madec et al., 1988-2013), POM (Mellor, 1991-2013)**
-
- **Black Sea modeling: Korotaev et al., (1991-2013), Stanev (2005), Ivanov et al., (2007, 2012), Trukhchev et al., (1980-2012)**
- **INM (Marchuk, Zalesny, Diansky, Moshonkin, Rusakov, Gusev et al., (1980 - 2013)**

Multicomponent Splitting Method
(Marchuk , Yanenko, Samarskii et al)

- **Splitting method is a methodological basis for the construction and treatment of the complicated systems**
- **Symmetrized form of govening equations**
- **Energy conserving space approximation (V.I. Lebedev grid)**
- **Splitting into series of nonnegative subproblems**
- **Adjoint system for each subproblem. Adjoint model consists of the respective subsystems adjoint to the split subsystems of the forward model**
- **Implicit schemes and exact solutions**

Splitting method. Solution of complicated forward and inverse problems

- **Stable implicit schemes**

$$\frac{\partial \boldsymbol{\varphi}}{\partial t} + \left(A_1 \boldsymbol{\varphi} + A_2 \boldsymbol{\varphi} + \ldots + A_n \boldsymbol{\varphi}\right) = 0, \quad \boldsymbol{\varphi}(0) = \boldsymbol{\varphi}^0, \qquad A_i \geq 0$$

$$\begin{cases} \dfrac{\partial \boldsymbol{\varphi}^{j+1/n}}{\partial t} + A_1 \boldsymbol{\varphi}^{j+1/n} = 0, & \boldsymbol{\varphi}^{j+1/n} = \boldsymbol{\varphi}^0 \\ ---------------------- & \\ \dfrac{\partial \boldsymbol{\varphi}^{j+1}}{\partial t} + A_n \boldsymbol{\varphi}^{j+1} = 0, & \boldsymbol{\varphi}^{j+1} = \boldsymbol{\varphi}^{j+(n-1)/n} \end{cases}$$

$$A_i \geq 0 \; !$$

Splitting by physical processes. Stage I: convection-diffusion

$$\frac{du}{dt} = D_u u$$

$$\frac{dv}{dt} = D_v v$$

$$\frac{\partial Z_{\boldsymbol{\sigma}} T}{\partial t} + \frac{1}{r_y r_x}\frac{\partial}{\partial x}\left(Z_{\boldsymbol{\sigma}} r_y u T\right) + \frac{1}{r_x r_y}\frac{\partial}{\partial y}\left(Z_{\boldsymbol{\sigma}} r_x v T\right) + \frac{\partial}{\partial \boldsymbol{\sigma}}(\boldsymbol{\omega} T) = D_T\, T$$

$$\frac{\partial Z_{\boldsymbol{\sigma}} S}{\partial t} + \frac{1}{r_y r_x}\frac{\partial}{\partial x}\left(Z_{\boldsymbol{\sigma}} r_y u S\right) + \frac{1}{r_x r_y}\frac{\partial}{\partial y}\left(Z_{\boldsymbol{\sigma}} r_x v S\right) + \frac{\partial}{\partial \boldsymbol{\sigma}}(\boldsymbol{\omega} S) = D_S\, S$$

Splitting by physical processes.
Stage II: adaptation of density and velocity fields

$$\frac{\partial u}{\partial t} - (l + \xi) v = -\frac{1}{\rho_0 r_x} \left[\frac{\partial}{\partial x} \left(p - \frac{g}{2} \rho Z \right) + \frac{g}{2} \left(\rho \frac{\partial Z}{\partial x} - Z \frac{\partial \rho}{\partial x} \right) \right]$$

$$\frac{\partial v}{\partial t} + (l + \xi) u = -\frac{1}{\rho_0 r_y} \left[\frac{\partial}{\partial y} \left(p - \frac{g}{2} \rho Z \right) + \frac{g}{2} \left(\rho \frac{\partial Z}{\partial y} - Z \frac{\partial \rho}{\partial y} \right) \right]$$

$$\frac{\partial}{\partial \sigma} \left(p - \frac{g}{2} \rho Z \right) = \frac{g}{2} \left(\rho \frac{\partial Z}{\partial \sigma} - Z \frac{\partial \rho}{\partial \sigma} \right)$$

$$-\frac{\partial \zeta}{\partial t} + \frac{1}{r_x r_y} \left[\frac{\partial}{\partial x} (Z_\sigma r_y u) + \frac{\partial}{\partial y} (Z_\sigma r_x v) \right] + \frac{\partial \omega}{\partial \sigma} = 0$$

$$\frac{\partial Z_\sigma T}{\partial t} = 0 \qquad \frac{\partial Z_\sigma S}{\partial t} = 0 \qquad \rho = \tilde{\rho}(T + \bar{T}, S + \bar{S}, p) - \tilde{\rho}(\bar{T}, \bar{S}, \rho_0 g Z)$$

Splitting by space coordinates

$$\frac{\partial Z_{\sigma}\varphi}{\partial t}+\frac{1}{2r_y r_x}\left[\frac{\partial}{\partial x}\left(Z_{\sigma}ur_y\varphi\right)+Z_{\sigma}ur_y\frac{\partial\varphi}{\partial x}\right]-D_{x,x}\varphi=D_{x,\sigma}\varphi$$

$$\frac{\partial Z_{\sigma}\varphi}{\partial t}+\frac{1}{2r_y r_x}\left[\frac{\partial}{\partial y}\left(Z_{\sigma}vr_x\varphi\right)+Z_{\sigma}vr_x\frac{\partial\varphi}{\partial y}\right]-D_{y,y}\varphi=D_{y,\sigma}\varphi$$

$$\frac{\partial Z_{\sigma}\varphi}{\partial t}+\frac{1}{2}\left[\frac{\partial}{\partial\sigma}\left(\omega\varphi\right)+\omega\frac{\partial\varphi}{\partial\sigma}\right]=D_{\sigma,\sigma}\varphi$$

Splitting method – parallel computing

- **Solution with respect to time. Explicit scheme**

$$\frac{\partial \varphi}{\partial t} + u\frac{\partial \varphi}{\partial x} = 0, \quad \Rightarrow \frac{\partial \varphi}{\partial t} + \frac{u}{3}\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial x}\right) = 0$$

$$\begin{cases} \dfrac{\varphi^{j+1/3} - \varphi^{j}}{\tau} + \dfrac{u}{3}\dfrac{\partial \varphi^{j}}{\partial x} = 0, \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \\ \dfrac{\varphi^{j+1} - \varphi^{j+2/3}}{\tau} + \dfrac{u}{3}\dfrac{\partial \varphi^{j+2/3}}{\partial x} = 0 \end{cases}$$

$$\frac{\varphi^{j+1} - \varphi^{j}}{\tau} + u\frac{\partial \varphi^{j}}{\partial x} + \frac{\tau^{2}u^{3}}{27}\frac{\partial^{3}\varphi^{j}}{\partial x^{3}} = \frac{\tau u^{2}}{3}\frac{\partial^{2}\varphi^{j}}{\partial x^{2}}$$

Ocean modeling. New problems

- Our physical understanding of the sea and ocean circulation is built on observations, theory, and modeling
- With advances in computer power and numerical technique, realistic simulation with nonlinear ocean general circulation models became possible
- Numerical sea and ocean models are bringing new insights into the physical processes operating in the sea and will soon find applications in Earth system modeling, weather forecasting, and ecosystem simulation

4D-var data assimilation technique

(Marchuk, Penenko, 1978; Agoshkov et al., 2007-2013)

- **4D-Var data assimilation method is applied in oceanography to solve inverse problems**
- **It is used to find a set of control variables, which minimize the norm of distance between observations and model predictions (cost function)**
- **Using adjoint equation method the gradient of the cost function is computed and optimal control method is implemented to solve problems arising in ocean modeling**

Ocean computing system

- Main component of the computing system is the forward σ-coordinate model of the World Ocean splitted by physical processes and coordinates
- Second component is an adjoint model. It consists of the respective subsystems adjoint to the split subsystems of the forward model
- Third component is an algorithm for functional minimization describing a measure of the misfit between data and model fields

Variational initialization method (Marchuk, Agoshkov, Shutyaev, Parmuzin, Zalesny, 1995-2013)

$$J = \underbrace{\underbrace{\frac{1}{\Sigma_0}\int_{\Sigma_0} \boldsymbol{\alpha}^0 \left(\boldsymbol{\varphi}^0 - \boldsymbol{\varphi}^0_{data}\right)^2 d\boldsymbol{\Sigma}_0 + \frac{1}{\Sigma_1}\int_{\Sigma_1} \boldsymbol{\beta}^0 \left(\boldsymbol{\varphi}^0 - \boldsymbol{\varphi}^0_{\mathrm{mod}el}\right)^2 d\boldsymbol{\Sigma}_1}_{3D-VAR} + \frac{1}{\left(\bar{t} - \boldsymbol{\tau}\right)}\int_{\boldsymbol{\tau}}^{\bar{t}} \frac{1}{\Sigma_2}\int_{\Sigma_2} \boldsymbol{\alpha}\left(\boldsymbol{\varphi} - \boldsymbol{\varphi}_{data}\right)^2 d\boldsymbol{\Sigma}_2\, dt}_{4D-VAR}$$

$$\boldsymbol{\varphi} \in \begin{cases} \dfrac{\partial \boldsymbol{\varphi}}{\partial t} + A(\boldsymbol{\varphi}) = f \\ \boldsymbol{\varphi}^0 \in R_N \end{cases} \qquad \begin{cases} -\dfrac{\partial \boldsymbol{\varphi}^*}{\partial t} + A'^*\left(\boldsymbol{\varphi}^*\right) + \tilde{\boldsymbol{\alpha}}\left(\boldsymbol{\varphi} - \boldsymbol{\varphi}_{data}\right) = 0 \\ \boldsymbol{\varphi}^*\left(\bar{t}\right) = 0 \end{cases}$$

$$grad\, J = -\boldsymbol{\varphi}^*(0) + \tilde{\boldsymbol{\beta}}\left(\boldsymbol{\varphi}^0 - \boldsymbol{\varphi}^0_{\mathrm{mod}el}\right)$$

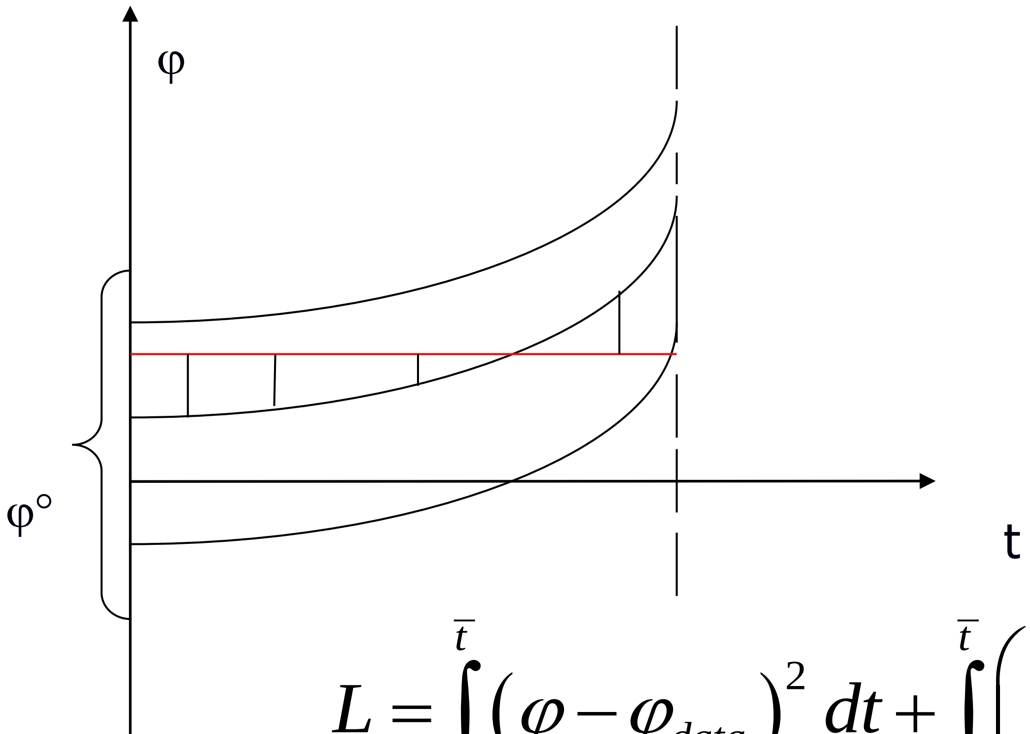
Initialization problem

$$\frac{\partial \boldsymbol{\varphi}}{\partial t} + A\boldsymbol{\varphi} - f = 0$$

$$\boldsymbol{\varphi}(0) = ?$$

$$J = \int_0^{\bar{t}} \left(\boldsymbol{\varphi} - \boldsymbol{\varphi}_{data}\right)^2 dt \rightarrow \min$$

$$\boldsymbol{\varphi} \in \boldsymbol{\Phi}$$



$$L = \int_0^{\bar{t}} \left(\varphi - \varphi_{data}\right)^2 dt + \int_0^{\bar{t}} \left(\frac{\partial \varphi}{\partial t} + A\varphi - f, \varphi^* \right) dt \rightarrow \min$$

$$\frac{\partial L}{\partial \boldsymbol{\varphi}^*} = 0 \qquad \frac{\partial L}{\partial \boldsymbol{\varphi}} = 0$$

Nonlinear optimality system. Boundary-value problem in [0,t] x Ω

$$v - v\left(\frac{\partial \rho(T)}{\partial z}\right) = 0,$$

$$\frac{\partial T}{\partial t} - \frac{\partial}{\partial z} v \frac{\partial T}{\partial z} = 0,$$

$$-\frac{\partial T^*}{\partial t} - \frac{\partial}{\partial z} v \frac{\partial T^*}{\partial z} + \alpha\left(T - \hat{T}\right) = -\frac{\partial \rho(T)}{\partial T} \cdot \frac{\partial}{\partial z}\left(v' \cdot v^*\right),$$

$$v^* + \frac{\partial T}{\partial z}\frac{\partial T^*}{\partial z} = 0$$

$$T^* = 0, \qquad for \qquad t = t_1, \quad t = t_0$$

Linear 4D problem for adjoint function

$$\left(\frac{\partial}{\partial t}+A\right)\alpha^{-1}\left(-\frac{\partial}{\partial t}+A^{*}\right)\varphi^{*}=\left(\frac{\partial}{\partial t}+A\right)\varphi_{data}$$

$$\varphi^{*}=0, \quad for \quad t=t_{0}, \quad t=t_{1}$$

Linear 4D problem for adjoint function. Time scheme

$$\frac{\varphi^{j+1} - \varphi^{j}}{\tau} + A_1 \varphi^{j+1} + A_0 \varphi^{j} = 0$$

$$(E + \tau A_1)(E - \tau A_0)\varphi^{*\,j-1} - \left[(E + \tau A_1)^2 + (E - \tau A_0)^2\right]\varphi^{*\,j} +$$

$$+ (E + \tau A_1)(E - \tau A_0)\varphi^{*\,j+1} = f$$

Two points in time boundary-value problems for adjoint function

$$\frac{\partial^2 \varphi^*}{\partial t^2} - AA^* \varphi^* + \left(A - A^*\right)\varphi^* = f$$

Implicit scheme

$$\left[E + \frac{\tau}{2}\left(A + A^*\right)\right]\frac{\varphi^{*\,j+1} - 2\varphi^{*\,j} + \varphi^{*\,j-1}}{\tau^2} - AA^*\varphi^{*\,j} + \left(A - A^*\right)\frac{\varphi^{*\,j+1} - \varphi^{*\,j-1}}{2\tau} = f$$

Explicit scheme

$$\left[E - \frac{\tau}{2}\left(A + A^*\right)\right]\frac{\varphi^{*\,j+1} - 2\varphi^{*\,j} + \varphi^{*\,j-1}}{\tau^2} - AA^*\varphi^{*\,j} + \left(A - A^*\right)\frac{\varphi^{*\,j+1} - \varphi^{*\,j-1}}{2\tau} = f$$

Numerical experiments. INM Ocean Model (Marchuk et al., 2013)

- **σ-coordinate system**

- **Evolutional form of governing equations**
- **Symmetrization**

- **Splitting into energy conserving/ nonnegative subsystems**

- **Adjoint subsystems. Implicit schemes**

PE σ-coordinate ocean dynamics equations

$$\boldsymbol{\sigma} = \frac{Z-\zeta}{H-\zeta}$$

$$\frac{du}{dt} - lv = -\frac{1}{\boldsymbol{\rho}_0 r_x}\frac{\partial p}{\partial x} + \frac{g}{\boldsymbol{\rho}_0 r_x}\frac{\partial Z}{\partial x}\boldsymbol{\rho} + \boldsymbol{\Lambda}_u u$$

$$\frac{dv}{dt} + lu = -\frac{1}{\boldsymbol{\rho}_0 r_y}\frac{\partial p}{\partial y} + \frac{g}{\boldsymbol{\rho}_0 r_y}\frac{\partial Z}{\partial y}\boldsymbol{\rho} + \boldsymbol{\Lambda}_v v$$

$$\frac{\partial p}{\partial \boldsymbol{\sigma}} = gZ_{\boldsymbol{\sigma}}\boldsymbol{\rho}$$

$$-\frac{\partial \boldsymbol{\zeta}}{\partial t} + \frac{1}{r_x r_y}\left[\frac{\partial}{\partial x}\left(Z_{\boldsymbol{\sigma}} r_y u\right) + \frac{\partial}{\partial y}\left(Z_{\boldsymbol{\sigma}} r_x v\right)\right] + \frac{\partial \boldsymbol{\omega}}{\partial \boldsymbol{\sigma}} = 0$$

$$\frac{d}{dt}\left(Z_{\boldsymbol{\sigma}} T\right) = \boldsymbol{\Lambda}_T T \qquad \frac{d}{dt}\left(Z_{\boldsymbol{\sigma}} S\right) = \boldsymbol{\Lambda}_S S$$

$$\boldsymbol{\rho} = \boldsymbol{\rho}(T,S,Z)$$

Total Energy Conservation Law

- **Considering PE ocean model without frictional and external forcers**
- **and multiplying equations by vector**

$$\left(\rho_0 u, \rho_0 v, \omega, p, -gZ\right)$$

- **we have**

$$\frac{\partial}{\partial t}\int_{\Sigma} \frac{\rho_0}{2} Z_{\sigma}\left(u^2 + v^2\right) d\boldsymbol{\Sigma} - \frac{\partial}{\partial t}\int_{\Sigma} g Z_{\sigma} Z \rho \, d\boldsymbol{\Sigma} = 0$$

Symmetrized form of ocean dynamics equations

$$\frac{du}{dt} - \tilde{l}\,v = -\frac{1}{\rho_0 r_x}\frac{\partial \tilde{p}}{\partial x} + \underbrace{\frac{g}{2\rho_0 r_x}\left(\rho\frac{\partial Z}{\partial x} - Z\frac{\partial \rho}{\partial x}\right)}_{1} + \Lambda_u u$$

$$\frac{dv}{dt} + \tilde{l}\,u = -\frac{1}{\rho_0 r_y}\frac{\partial \tilde{p}}{\partial y} + \underbrace{\frac{g}{2\rho_0 r_y}\left(\rho\frac{\partial Z}{\partial y} - Z\frac{\partial \rho}{\partial y}\right)}_{2} + \Lambda_v v$$

$$\frac{\partial \tilde{p}}{\partial \sigma} = \underbrace{\frac{g}{2}\left(\rho\frac{\partial Z}{\partial \sigma} - Z\frac{\partial \rho}{\partial \sigma}\right)}_{3}$$

$$-\frac{\partial \zeta}{\partial t} + \frac{1}{r_x r_y}\left[\frac{\partial}{\partial x}\left(D r_y u\right) + \frac{\partial}{\partial y}\left(D r_x v\right)\right] + \frac{\partial \omega}{\partial \sigma} = 0$$

$$D\frac{\partial \rho}{\partial t} + \frac{1}{2}\left\{\underbrace{\frac{1}{r_y r_x}\frac{\partial}{\partial x}\left(D u r_y \rho\right) + \frac{Du}{r_x}\frac{\partial \rho}{\partial x}}_{1} + \underbrace{\frac{1}{r_x r_y}\frac{\partial}{\partial y}\left(D v r_x \rho\right) + \frac{Dv}{r_y}\frac{\partial \rho}{\partial y}}_{2} + \underbrace{\frac{\partial}{\partial \sigma}(\omega \rho T) + \omega\frac{\partial \rho}{\partial \sigma}}_{3}\right\} = \Lambda \rho$$

Splitting method. Non-hydrostatic dynamics subsystem

$$\frac{du}{dt} = -\frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial x} \qquad \frac{d\text{v}}{dt} = -\frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial y}$$

$$\frac{dw}{dt} = -\frac{1}{\rho_0}\left(\frac{\partial \tilde{p}}{\partial z} - g\rho\right) \qquad \frac{\partial u}{\partial x} + \frac{\partial \text{v}}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\left(\frac{\partial \tilde{p}}{\partial z} - g\rho\right) = \frac{\partial}{\partial z}\left[\tilde{p} - \left(p_{atm} - g\rho_0\zeta + g\int_0^z \rho\, dz_1\right)\right]$$

$$p(t,x,y,z) = \tilde{p}(t,x,y,z) - \underbrace{\left[p_{atm} - g\rho_0\zeta(t,x,y) + g\int_0^z \rho(t,x,y,z_1)dz_1\right]}_{p_g}$$

Introduce σ- coordinate and ε - regularization

$$Z_\sigma \frac{\partial u}{\partial t} = \underbrace{g\, Z_\sigma \frac{\partial \zeta}{\partial x} - \frac{1}{\rho_0} Z_\sigma \frac{\partial p}{\partial x}}_{1} + \underbrace{\frac{1}{\rho_0} Z_x \frac{\partial p}{\partial \sigma}}_{4} + f_1$$

$$Z_\sigma \frac{\partial \mathrm{v}}{\partial t} = \underbrace{g\, Z_\sigma \frac{\partial \zeta}{\partial y} - \frac{1}{\rho_0} Z_\sigma \frac{\partial p}{\partial y}}_{2} + \underbrace{\frac{1}{\rho_0} Z_y \frac{\partial p}{\partial \sigma}}_{5} + f_2 \qquad Z_\sigma \frac{\partial w}{\partial t} = \underbrace{-\frac{1}{\rho_0} \frac{\partial p}{\partial \sigma}}_{3}$$

$$\varepsilon Z_\sigma \frac{\partial (p - \overline{p})}{\partial t} - \underbrace{\frac{\partial \zeta}{\partial t}}_{6} + \left[\underbrace{\frac{\partial}{\partial x}(Z_\sigma u)}_{1} + \underbrace{\frac{\partial}{\partial y}(Z_\sigma \mathrm{v})}_{2} \right] +$$

$$+ \frac{\partial}{\partial \sigma}\left[\underbrace{w}_{3} - \left(\underbrace{\frac{\partial Z}{\partial t}}_{6} + \underbrace{Z_x u}_{4} + \underbrace{Z_y \mathrm{v}}_{5} \right) \right] = 0.$$

NH dynamics subsystem into vertical coordinate

$$\frac{\partial u'}{\partial t} = \frac{1}{\rho_0 Z_\sigma}\left[\underbrace{Z_x \frac{\partial(p-\bar{p})}{\partial\sigma}}_{1} - \int_0^1 \left(Z_x \frac{\partial(p-\bar{p})}{\partial\sigma}\right) d\sigma\right]$$

$$\frac{\partial \mathrm{v}'}{\partial t} = \frac{1}{\rho_0 Z_\sigma}\left[\underbrace{Z_y \frac{\partial(p-\bar{p})}{\partial\sigma}}_{2} - \int_0^1 \left(Z_y \frac{\partial(p-\bar{p})}{\partial\sigma}\right) d\sigma\right]$$

$$\frac{\partial w}{\partial t} = -\underbrace{\frac{1}{\rho_0 Z_\sigma}\frac{\partial(p-\bar{p})}{\partial\sigma}}_{3} = 0$$

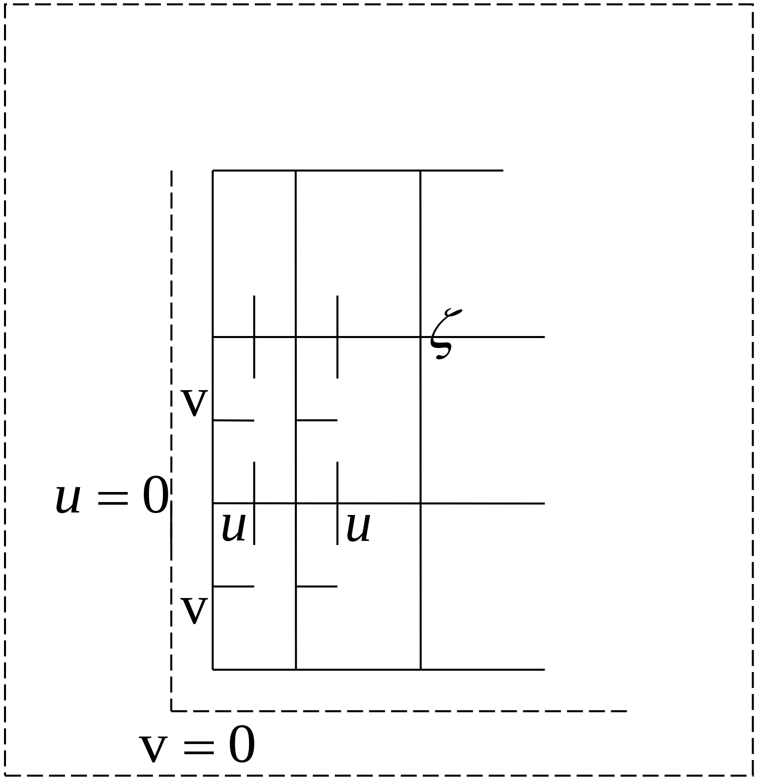
$$\varepsilon Z_\sigma \frac{\partial(p-\bar{p})}{\partial t} + \frac{\partial}{\partial\sigma}\left[\underbrace{w}_{3} - \left(Z_t + \underbrace{u Z_x}_{1} + \underbrace{\mathrm{v} Z_y}_{2}\right)\right] = 0.$$

Space approximation: FDM, FEM. Energy conservation law and lost of “locality”. Lumping.

$$\frac{\partial u}{\partial t} - l\,\mathrm{v} - \mathrm{g}\frac{\partial \zeta}{\partial \mathrm{x}} = 0$$

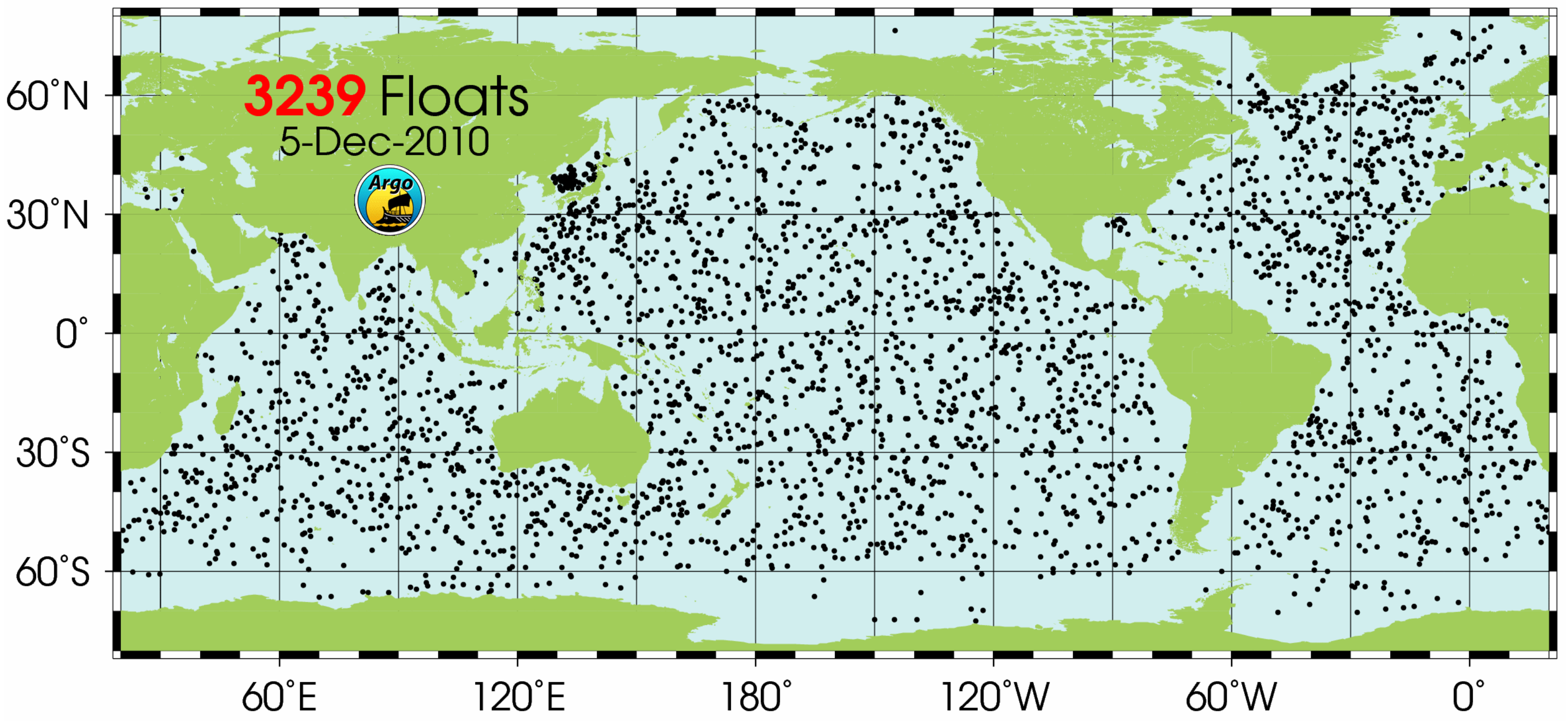
$$\frac{\partial \mathrm{v}}{\partial t} + l\,u - \mathrm{g}\frac{\partial \zeta}{\partial y} = 0, \quad (\vec{u}, \overline{n})\Big|_{\partial D} = 0$$

$$-\frac{\partial \zeta}{\partial t} + \frac{\partial H u}{\partial \mathrm{x}} + \frac{\partial H \mathrm{v}}{\partial y} = 0$$

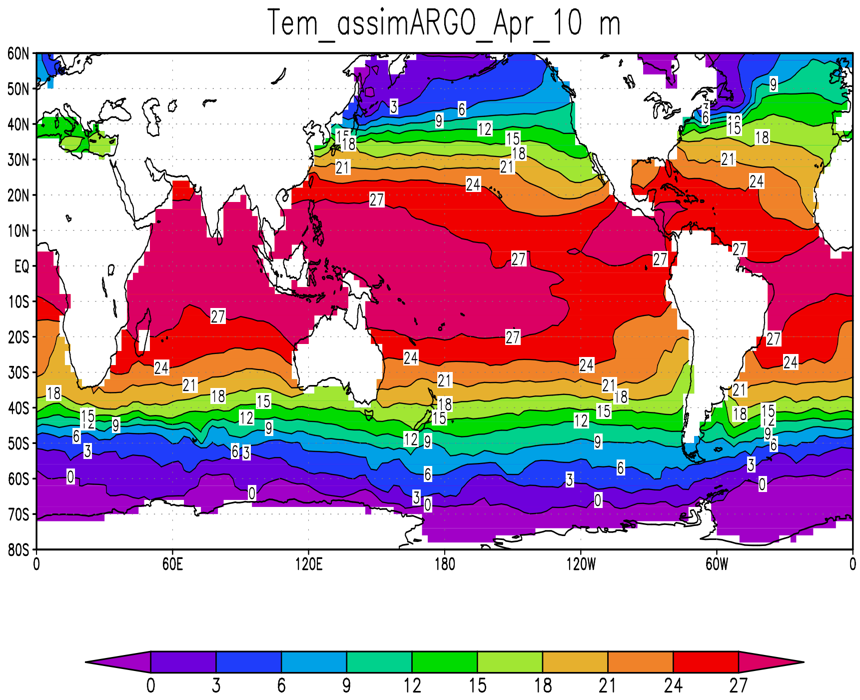


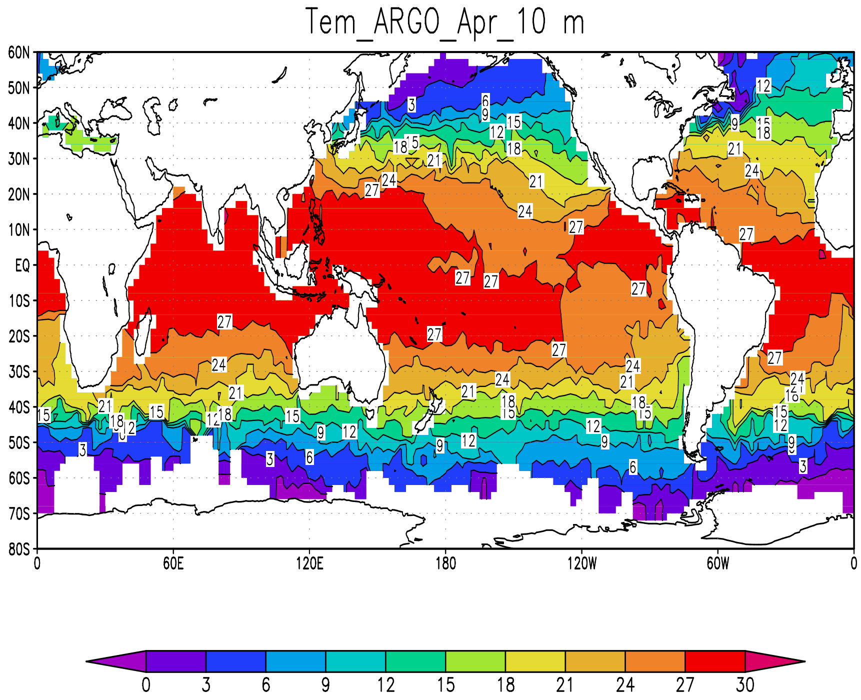
$$\begin{cases} \dfrac{\partial^2 \mathrm{v}}{\partial t^2} + l^2 \mathrm{v} - \mathrm{g}\left(\dfrac{\partial^2}{\partial t \partial y} - l\dfrac{\partial}{\partial x}\right)\zeta = 0 \\ \dfrac{\partial^2 \zeta}{\partial t^2} - \dfrac{\partial}{\partial x} gH \dfrac{\partial \zeta}{\partial x} - \left(\dfrac{\partial^2}{\partial t \partial y} + \dfrac{\partial}{\partial x} l\right) H \mathrm{v} = 0 \end{cases}$$

World Ocean. 4D VAR assimilation of Argo Data

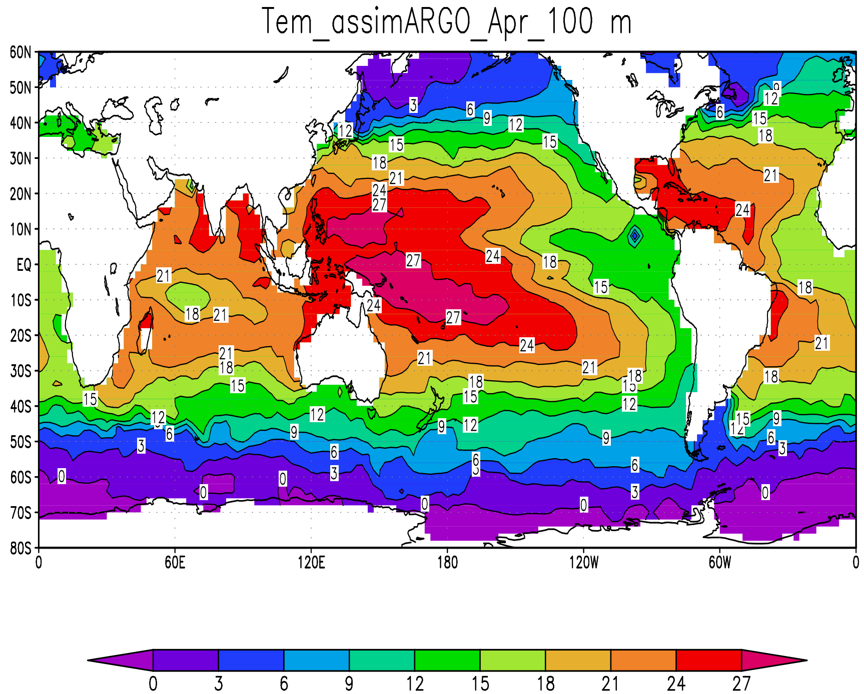


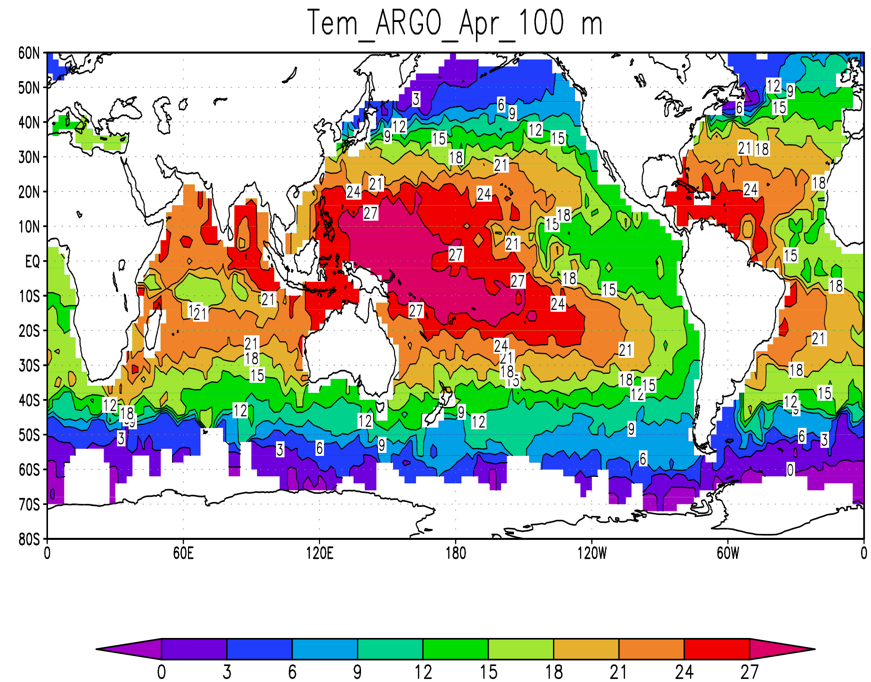
4D VAR World Ocean initialization. Temperature at 10 m, April 2008:
optimal solution (left), Argo data (Zakharova, right)



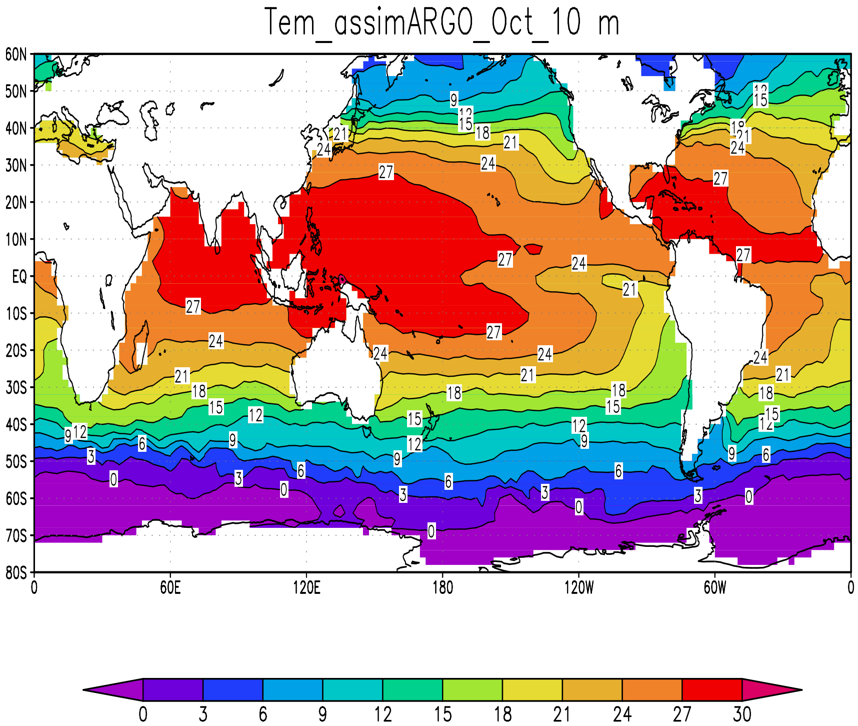


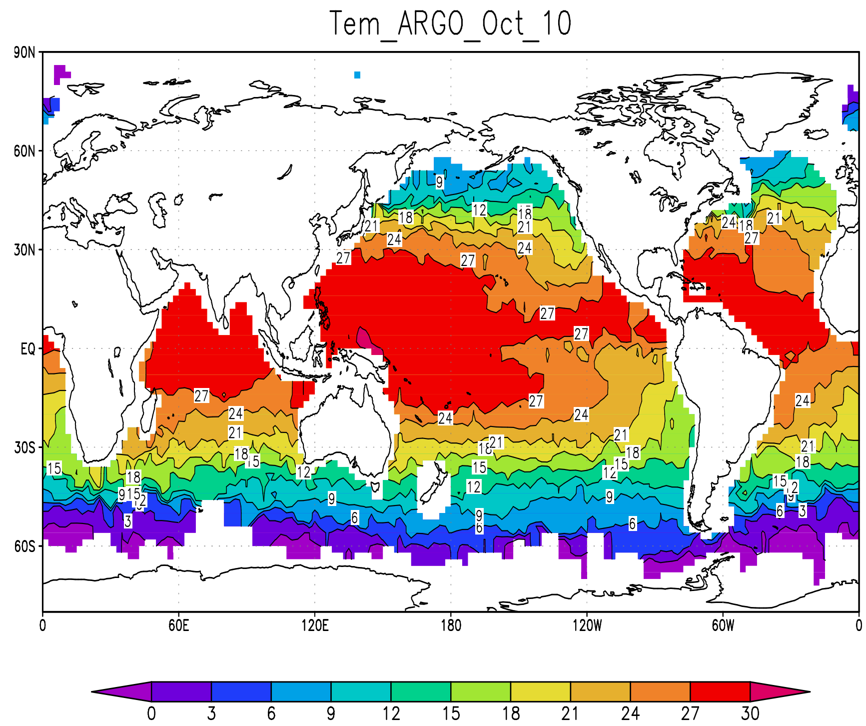
4D VAR World Ocean initialization. Temperature at 100 m, April 2008: optimal solution (left), Argo data (Zakharova, right)



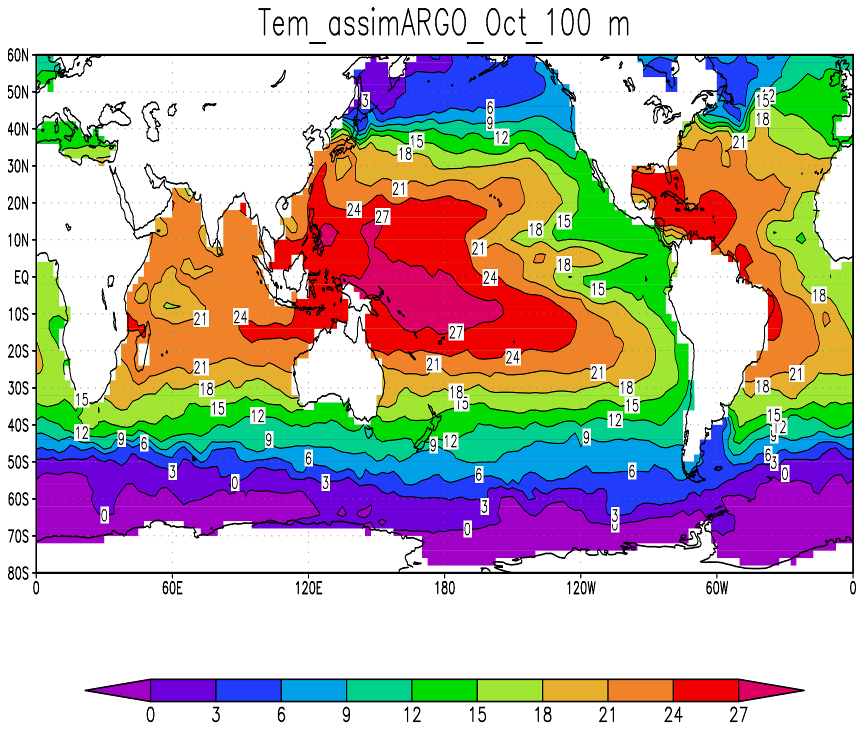


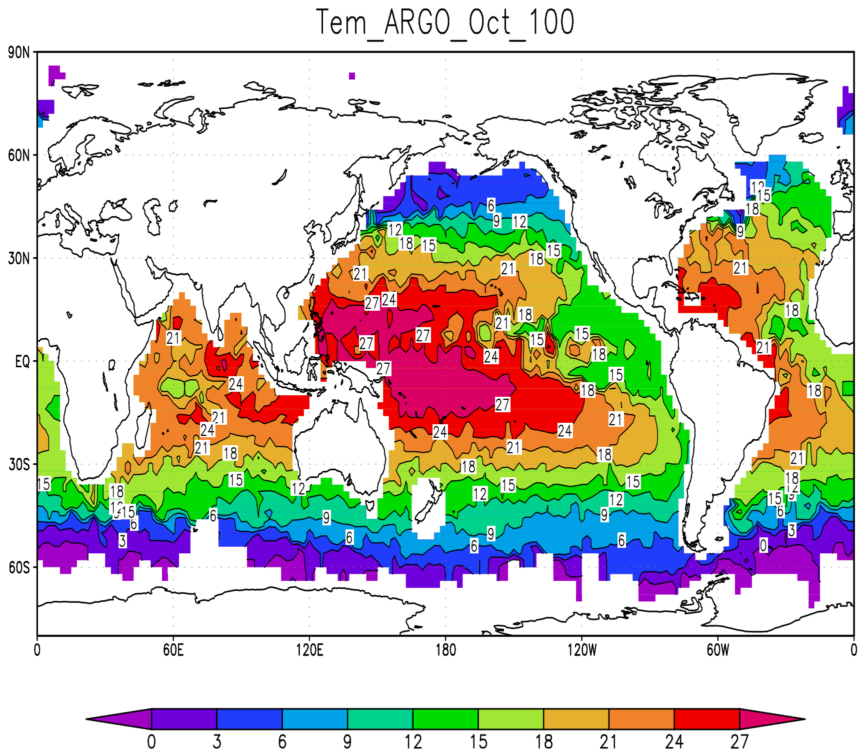
4D VAR World Ocean initialization. Temperature at 10 m, October 2008: optimal solution (left), Argo data (Zakharova, right)



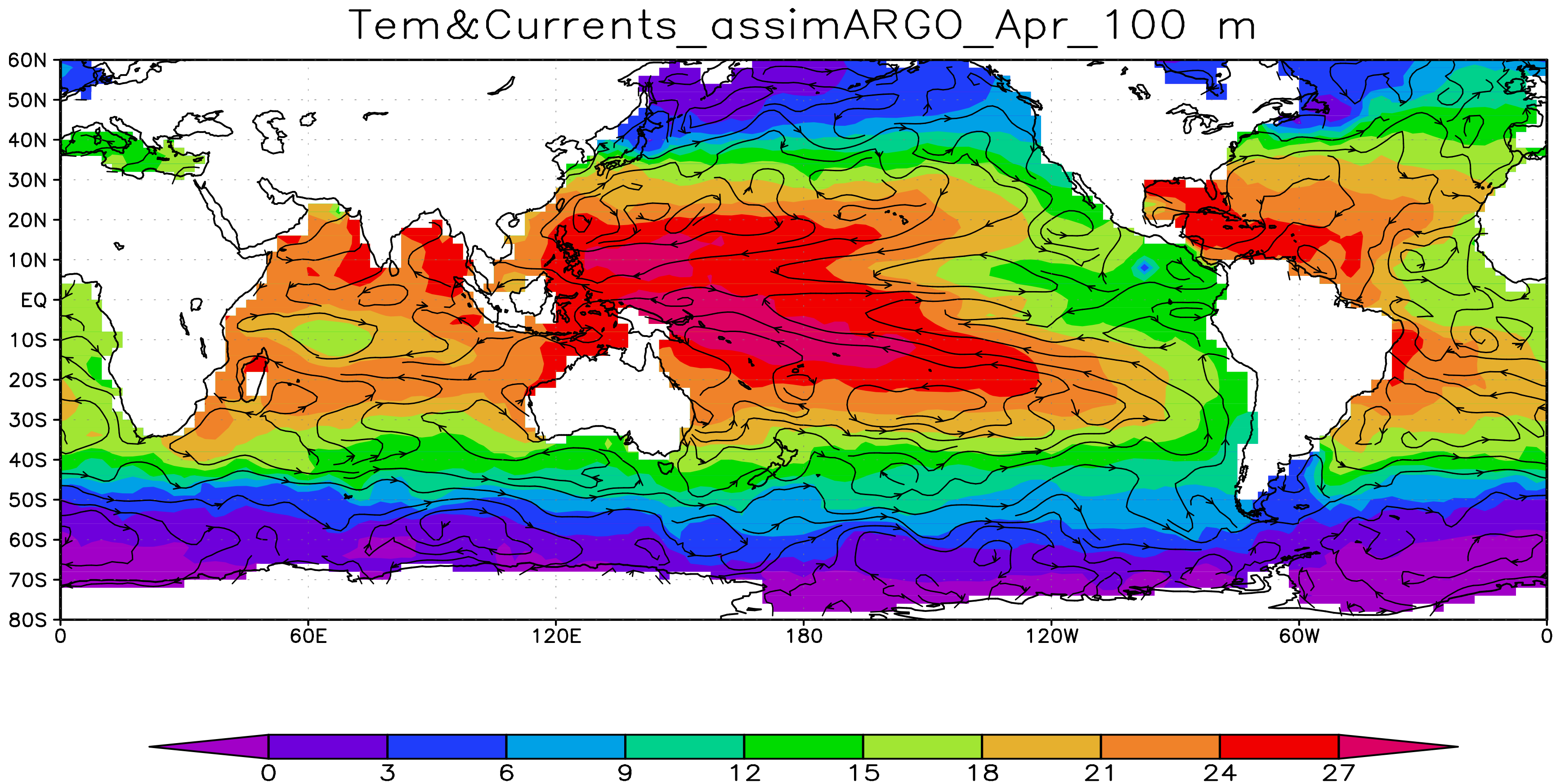


4D VAR World Ocean initialization. Temperature at 100 m, October 2008: optimal solution (left), Argo data (Zakharova, right)

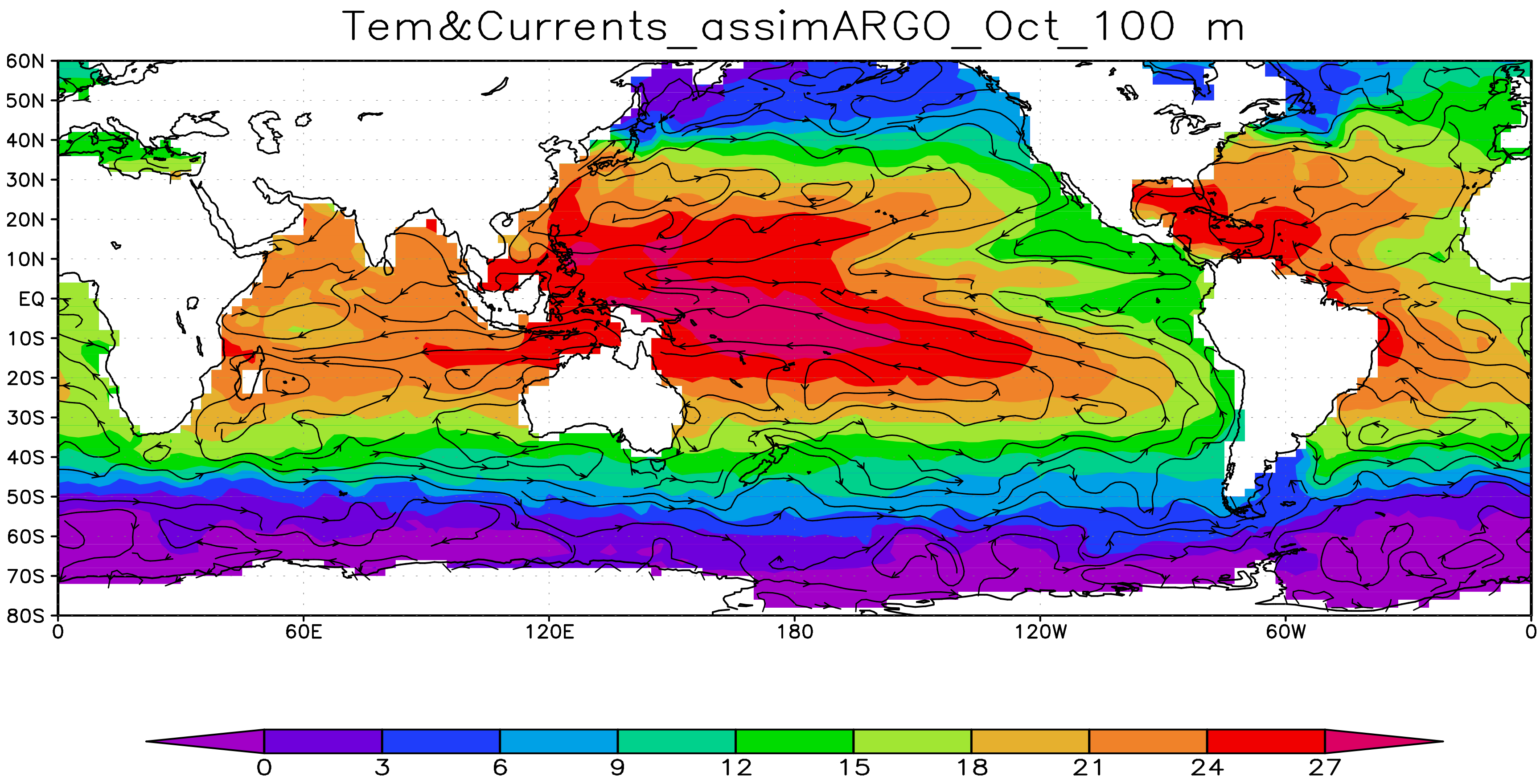




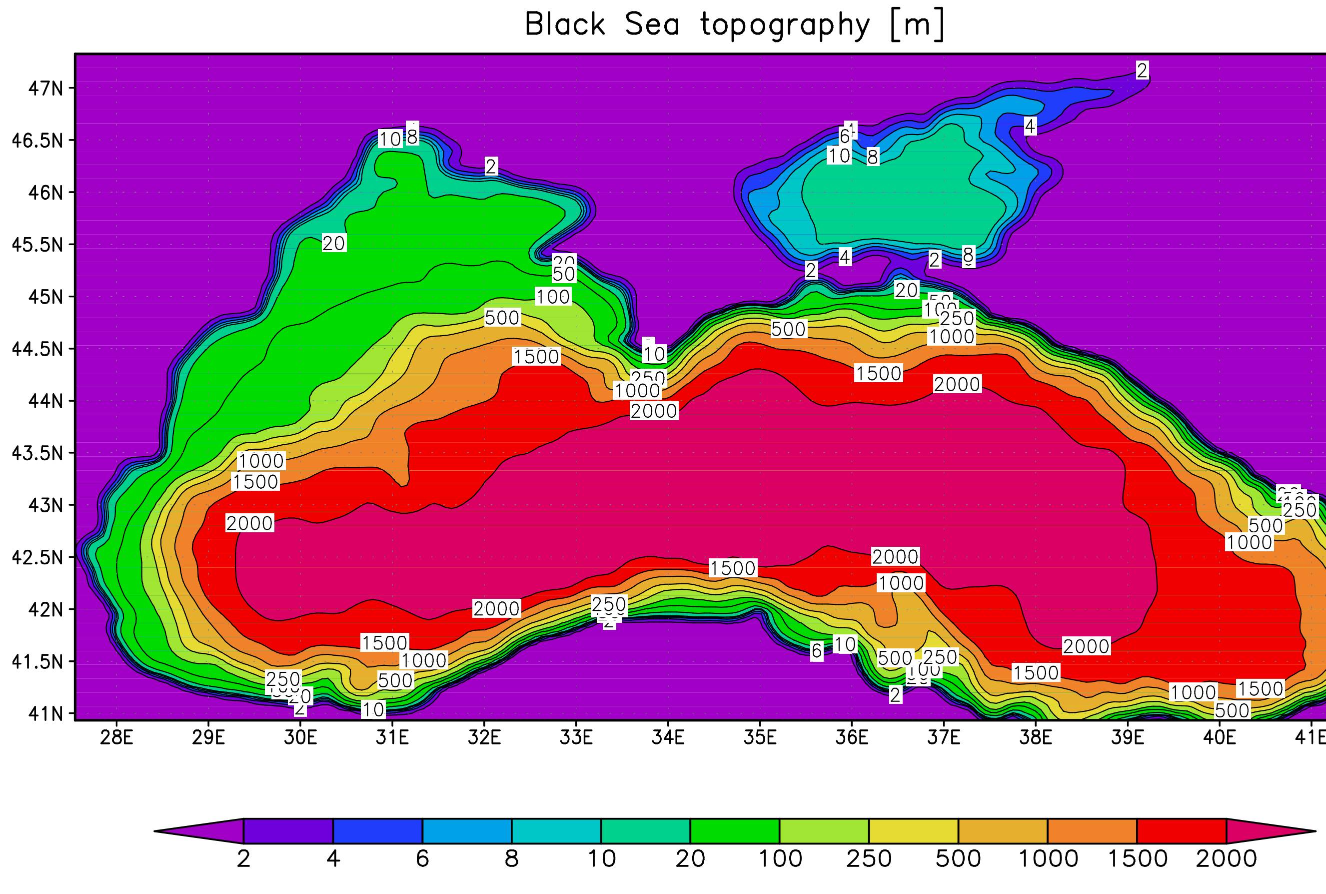
4D VAR World Ocean initialization. Temperature and currents at 100 m, April 2008. Optimal solution



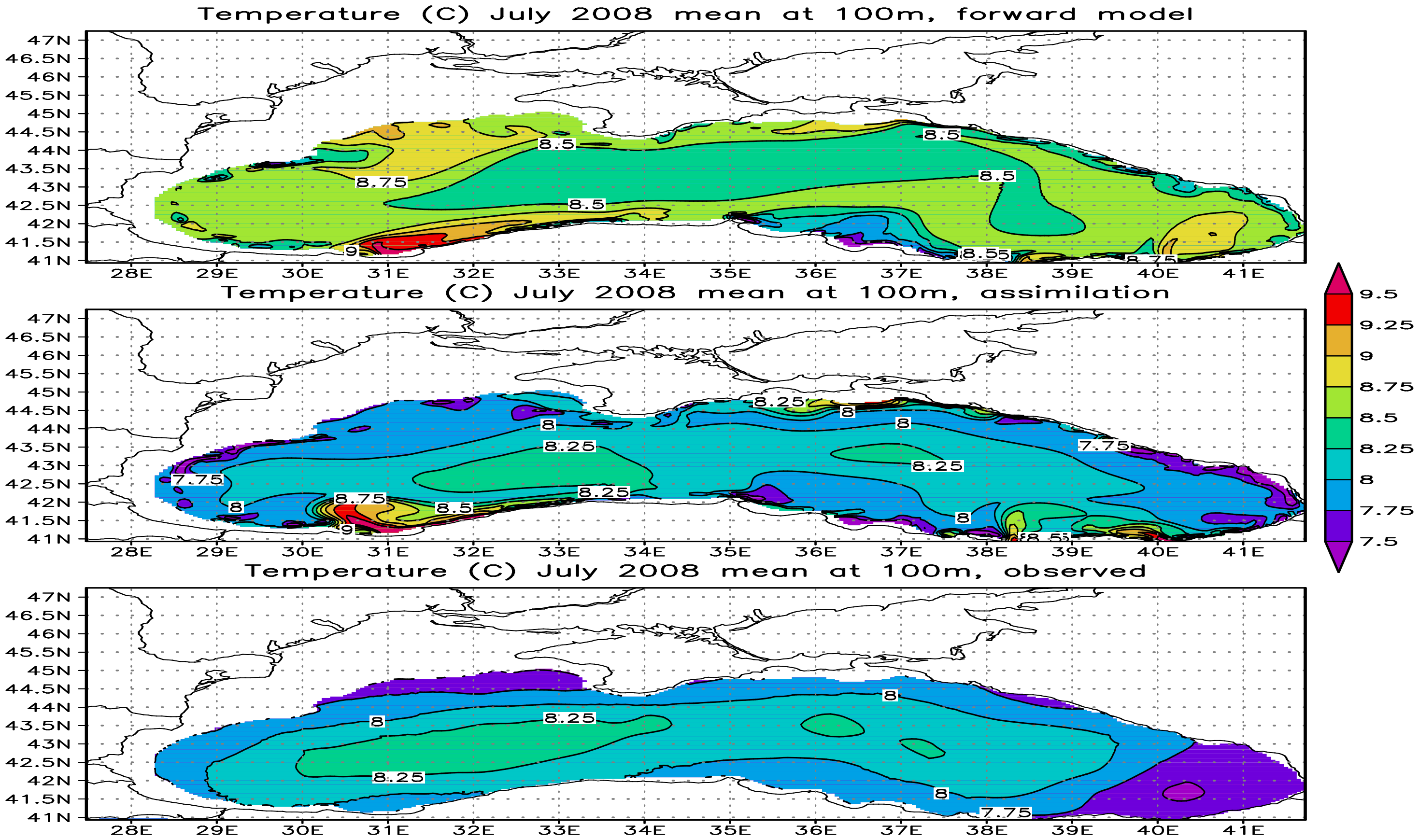
4D VAR World Ocean initialization. Temperature and currents at 100 m, October 2008. Optimal solution



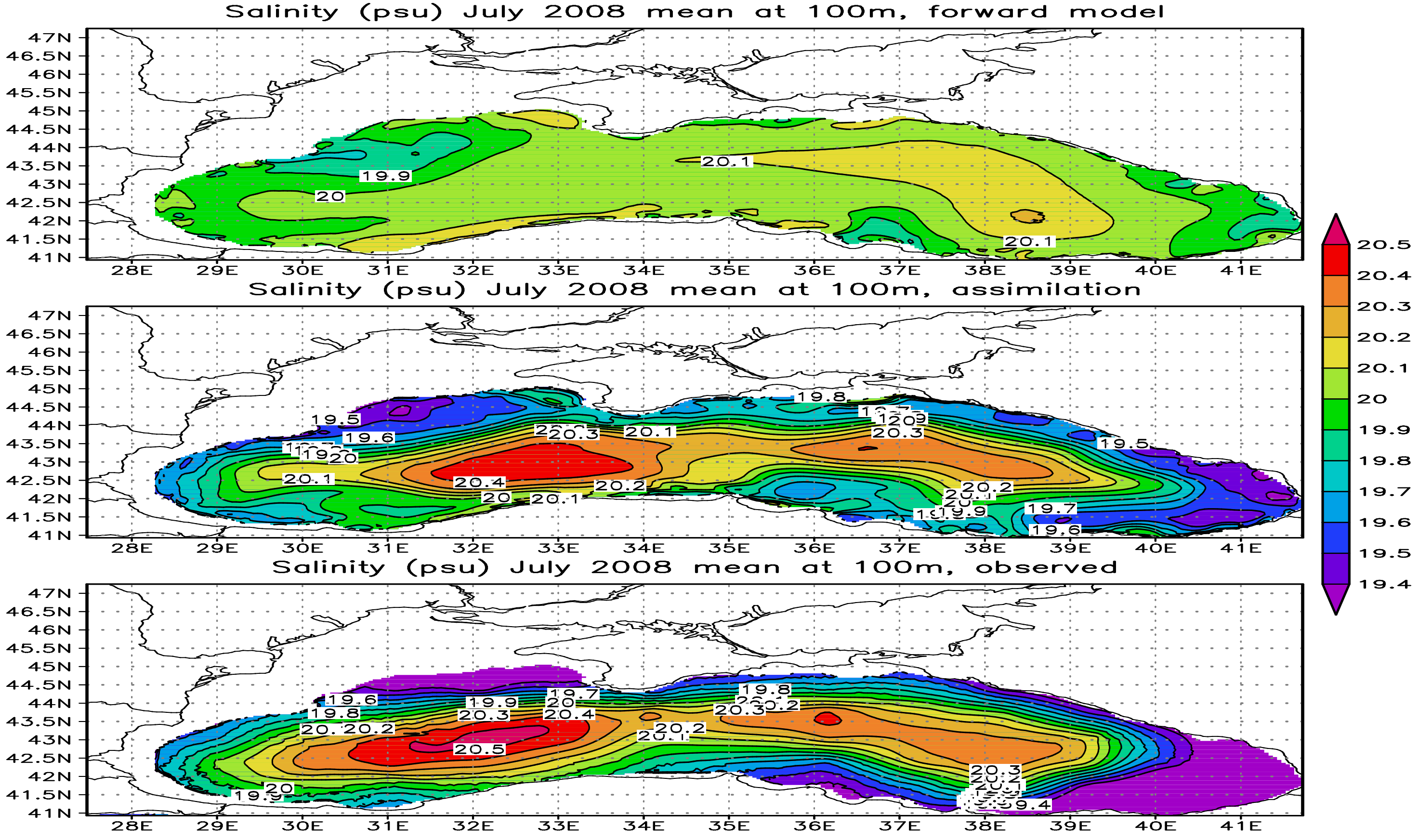
Black Sea model



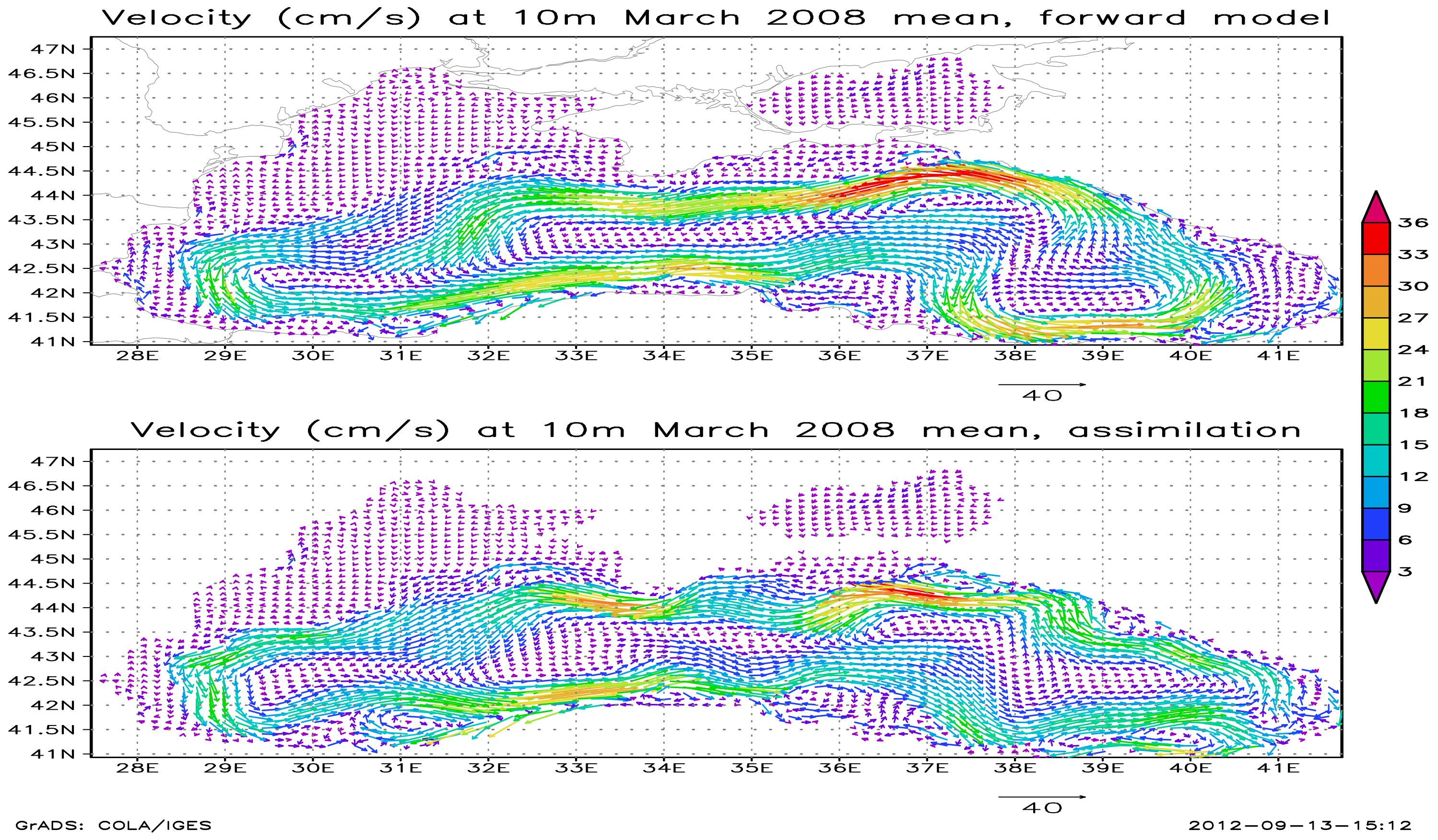
Monthly mean temperature: forward model, data assimilation model, observed data (top to bottom). July, 100 m



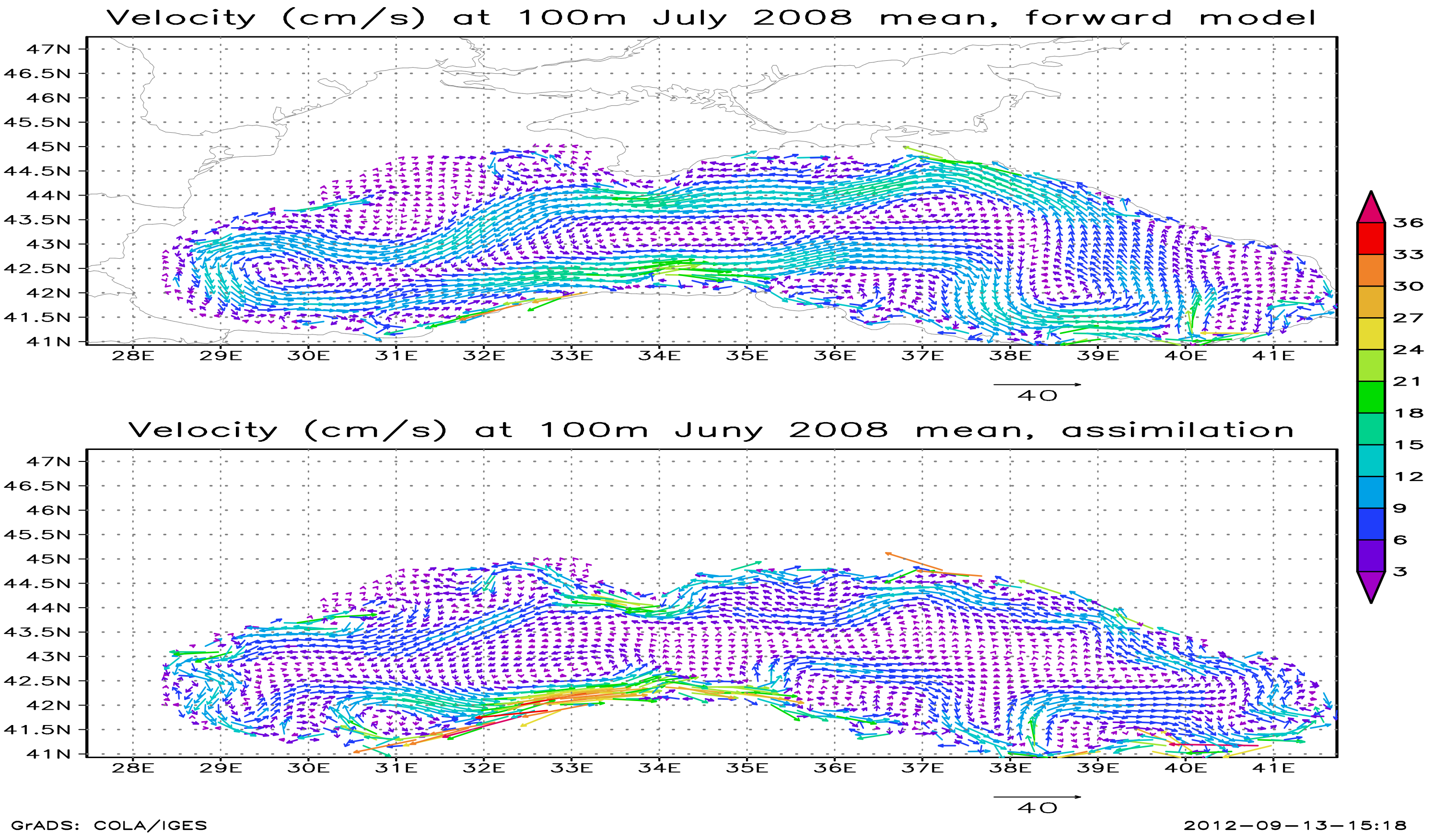
Monthly mean salinity: forward model, data assimilation model, observed data (top to bottom). July, 100 m



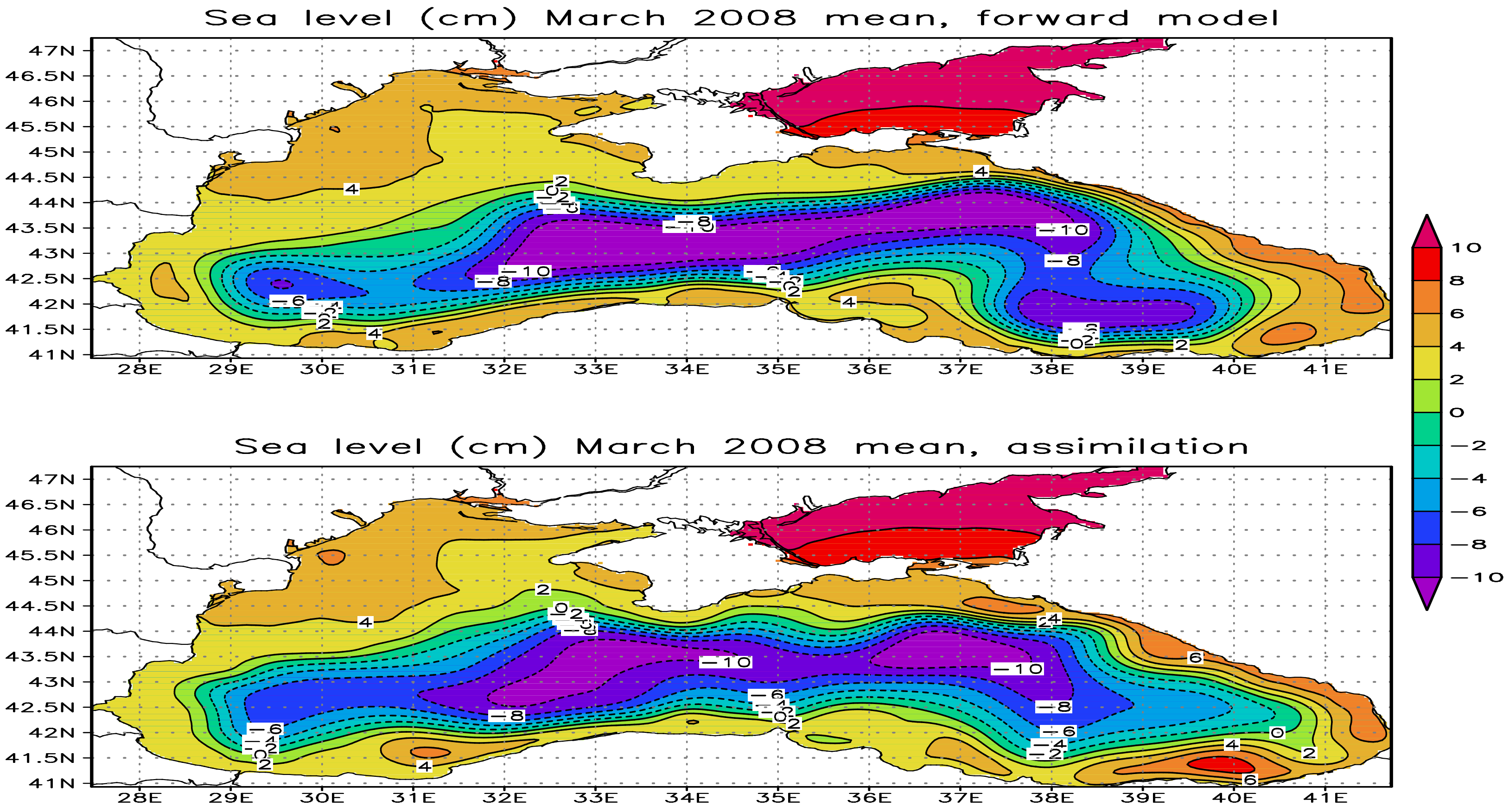
Monthly mean velocities: forward model, data assimilation model (top to bottom). March, 10 m



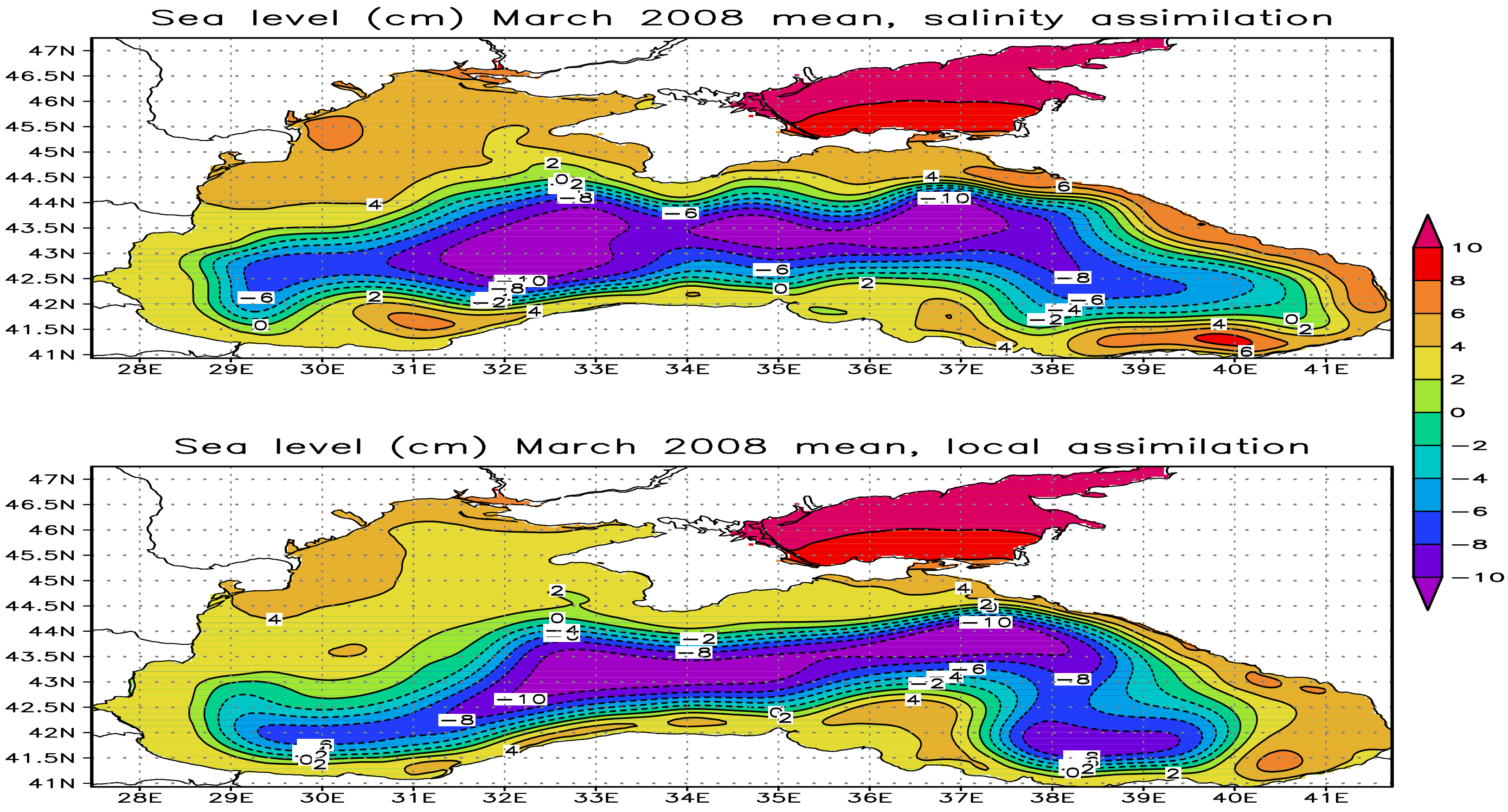
Monthly mean velocities: forward model, data assimilation model (top to bottom). July, 100 m



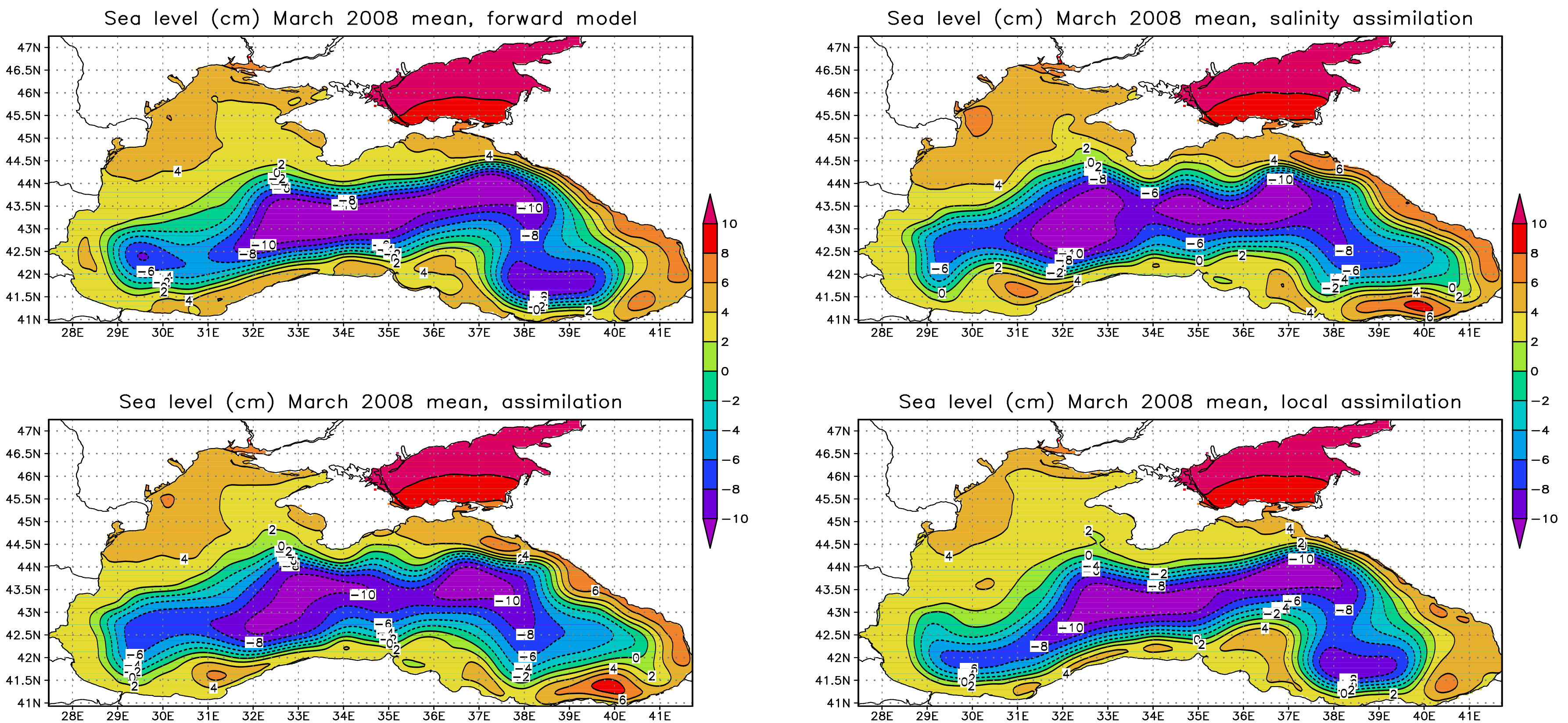
Monthly mean SSH: forward model, T, S data assimilation model (top to bottom). March



Monthly mean SSH: global salinity assimilation, T, S local assimilation (top to bottom). March



Monthly mean SSH: forward model (top left), T, S global assimilation (bottom left), S global (top right), T, S local (bottom right). March



Conclusions

- Splitting numerical technique for the solution of the prognostic and 4D-VAR ocean data assimilation problem is constructed
- As a result of splitting, a rather simple subsystems of the forward and adjoint equations are solved at each separate stage
- Adjoint model consists of the respective subsystems adjoint to the split subsystems of the forward model
- The method is the constructive basis for the INM modular computing system of simulation and initialization of the World Ocean hydrographic fields

Unresolved problems and future directions

- **Theory. Complicated equation of state. Nonlinear problems: sea level dynamics; non-hydrostatic equations; deep convection parameterization etc.**
- **High performance parallel computing. Unstructured adaptive meshes. Explicit-implicit algorithms with variable time step.**
- **New inverse problems and 4D VAR algorithms. Model calibration, parameter identification, ocean monitoring systems, etc.**
- **Models. Eddy-resolving global ocean prediction. Turbulent processes parameterization and modeling.**

БЛАГОДАРЯ ВИ ЗА ВНИМАНИЕТО